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ENERGY TRANSFERS IN MHD TURBULENCE AND ITS APPLICATIONS TO DYNAMO

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In this paper, we describe mode-to-mode energy transfers and energy fluxes of MHD turbulence. These energy transfers are very useful for understanding turbulence dynamics, as well as for applications, such as dynamo. We illustrate how the energy fluxes provide valuable insights into the mechanism of growth of large-scale magnetic energy in dynamo.

Introduction. Turbulence is a highly intractable problem, mostly due to the strong nonlinearity. The most well-known result of hydrodynamic turbulence was obtained by Kolmgorov [1]. According to this theory, when the fluid is forced at large scales, a constant energy flux Π flows from large scales to small scales via an intermediate scale. At the intermediate scale, the energy spectrum is $E(k) = C_{\mathrm{K}} \Pi^{2/3} k^{-5/3}$, where C_{K} is the Kolmogorov's constant.

MHD turbulence is more complex than hydrodynamic turbulence due to the larger number of variables (velocity and magnetic fields) and parameters (viscosity and magnetic diffusivity). Also, the equations of MHD turbulence have four nonlinear terms that induce more kinds of energy fluxes. In this paper, we describe these fluxes for MHD turbulence. Also, these fluxes are best derived using "mode-to-mode energy transfers" [2, 3], which are also discussed in this paper.

The energy fluxes and other energy transfer diagnostics are very useful for understanding MHD turbulence and its applications to dynamo. For example, using these fluxes, it is possible to compute the sources of magnetic energy at large scales, which enhances the magnetic field at these scales. The aim of this paper is to give more details and illustrate various mode-to-mode (velocity-to-velocity, velocity-to-magnetic, magnetic-to-magnetic, and magnetic-to-velocity) energy and helicity transfers. We emphasize arguments in favour of our formalism which removes the ambiguity in shell-to-shell transfer definitions.

The paper is presented in the following way. In Sections 1 and 2, the dynamic equations and energy equations are derived, respectively. The mode-to-mode energy transfers as well as their physical interpretations are described in Section 2. In Section 3, the energy fluxes in MHD turbulence are discussed. Section 4 contains an application of energy fluxes to dynamo. Conclusions are presented in Section 5.

1. Governing equations. The equations of incompressible magnetohydrodynamics are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + (\mathbf{B}\cdot\nabla)\mathbf{B} + \nu\nabla^2\mathbf{u} + \mathbf{F}_{\mathrm{ext}}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{u} + \eta\nabla^2\mathbf{B}, \tag{2}$$

$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{B} = 0, \tag{3}$$

where $\mathbf{u}, \mathbf{B}, p$ are, respectively, the velocity, the magnetic and pressure fields, ν is the kinematic viscosity, and η is the magnetic diffusivity. Note that the normalized pressure

p is the sum of hydrodynamic and magnetic pressures, and $\mathbf{B}$ could include a steady magnetic field (corresponding to the wavenumber $\mathbf{k}$=0). $\mathbf{F}_{\mathrm{ext}}$ is the external force field which is typically employed at large scales.

In the discrete Fourier space, the above equations are transformed to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(\mathbf{k}) + \mathbf{N}_u(\mathbf{k}) = -i\mathbf{k}p(\mathbf{k}) + \mathbf{F}_u(\mathbf{k}) - \nu k^2\mathbf{u}(\mathbf{k}) + \mathbf{F}_{\mathrm{ext}}(\mathbf{k}), \quad (4)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{B}(\mathbf{k}) + \mathbf{N}_B(\mathbf{k}) = \mathbf{F}_B(\mathbf{k}) - \eta k^2\mathbf{B}(\mathbf{k}), \quad (5)$$

$$\mathbf{k}\cdot\mathbf{u}(\mathbf{k}) = \mathbf{k}\cdot\mathbf{B}(\mathbf{k}) = 0, \quad (6)$$

where the nonlinear terms are

$$\mathbf{N}_u(\mathbf{k}) = i\sum_{\mathbf{p}}\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\mathbf{u}(\mathbf{p}); \quad \mathbf{N}_B(\mathbf{k}) = i\sum_{\mathbf{p}}\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\mathbf{B}(\mathbf{p}), \quad (7)$$

$$\mathbf{F}_u(\mathbf{k}) = i\sum_{\mathbf{p}}\{\mathbf{k}\cdot\mathbf{B}(\mathbf{q})\}\mathbf{B}(\mathbf{p}); \quad \mathbf{F}_B(\mathbf{k}) = i\sum_{\mathbf{p}}\{\mathbf{k}\cdot\mathbf{B}(\mathbf{q})\}\mathbf{u}(\mathbf{p}), \quad (8)$$

where $\mathbf{k} = \mathbf{p} + \mathbf{q}$. The terms $\mathbf{N}_u(\mathbf{k}), \mathbf{N}_B(\mathbf{k})$ represent the advection of the velocity and magnetic fields by the velocity fluctuation $\mathbf{u}(\mathbf{q})$, whereas $\mathbf{F}_u$ is the Lorentz force, and $\mathbf{F}_B$ corresponds to the nonlinear term $(\mathbf{B}\cdot\nabla)\mathbf{u}$. The pressure is determined using

$$p(\mathbf{k}) = i\frac{1}{k^2}\mathbf{k}\cdot[\mathbf{N}_u(\mathbf{k}) - \mathbf{F}_u(\mathbf{k})]. \quad (9)$$

We assume that $\mathbf{F}_{\mathrm{ext}}(\mathbf{k})$ is divergdence-free.

We also define the modal kinetic and magnetic energies using the following formulas:

$$E_u(\mathbf{k}) = \frac{1}{2}|\mathbf{u}(\mathbf{k})|^2; \quad E_b(\mathbf{k}) = \frac{1}{2}|\mathbf{b}(\mathbf{k})|^2. \quad (10)$$

In the next section, the equations for the kinetic and magnetic energies are presented and formulas for the mode-to-mode energy transfers for MHD turbulence are derived.

2. Energy equations. In a real space, the equations for the kinetic energy density $E_u(\mathbf{r})$ and magnetic energy density $E_b(\mathbf{r})$ are as follows:

$$\frac{\partial}{\partial t}E_u(\mathbf{r}) + u_j\partial_j\left(\frac{1}{2}u_iu_i\right) = -\partial_j(u_jp) + [B_j\partial_jB_i]u_i - \nu\omega^2, \quad (11)$$

$$\frac{\partial}{\partial t}E_b(\mathbf{r}) + u_j\partial_j\left(\frac{1}{2}B_iB_i\right) = [B_j\partial_ju_i]B_i - \eta J^2, \quad (12)$$

where $\omega = \nabla\times\mathbf{u}$ is the vorticity, and $\mathbf{J} = \nabla\times\mathbf{B}$ is the current density. To derive the corresponding equations in the Fourier space, we perform a scalar product of Eq. (4) with $\mathbf{u}^*(\mathbf{k})$; the resulting equation and its complex conjugate are added to derive an equation for the modal kinetic energy. Similar operations for Eq. (5) with $\mathbf{B}^*(\mathbf{k})$ yield the equation for the modal magnetic energy. These equations are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_u(\mathbf{k}) &= -\Re[\mathbf{N}_u(\mathbf{k})\cdot\mathbf{u}^*(\mathbf{k})] + \Re[\mathbf{F}_u(\mathbf{k})\cdot\mathbf{u}^*(\mathbf{k})] \quad (13)\\ &= \sum_{\mathbf{p}}\Im\left[\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}(\mathbf{p})\cdot\mathbf{u}^*(\mathbf{k})\} - \{\mathbf{k}\cdot\mathbf{B}(\mathbf{q})\}\{\mathbf{B}(\mathbf{p})\cdot\mathbf{u}^*(\mathbf{k})\}\right],\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_B(\mathbf{k}) &= -\Re[\mathbf{N}_B(\mathbf{k})\cdot\mathbf{B}^*(\mathbf{k})] + \Re[\mathbf{F}_B(\mathbf{k})\cdot\mathbf{B}^*(\mathbf{k})] \quad (14)\\ &= \sum_{\mathbf{p}}\Im\left[\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{B}(\mathbf{p})\cdot\mathbf{B}^*(\mathbf{k})\} - \{\mathbf{k}\cdot\mathbf{B}(\mathbf{q})\}\{\mathbf{u}(\mathbf{p})\cdot\mathbf{B}^*(\mathbf{k})\}\right],\end{aligned}$$

where $\mathbf{k} = \mathbf{p} + \mathbf{q}$. For simplification, we set $\nu = \eta = 0$. Note that the pressure $p(\mathbf{k})$ disappears due to the incompressibility condition $\mathbf{k}\cdot\mathbf{u}(\mathbf{k}) = 0$.

Table 1. Summary of various combined energy transfers in MHD turbulence.

ET	Receiver mode	Giver modes
$S^{uu}(\mathbf{X}\|\mathbf{Y},\mathbf{Z})$	$\mathbf{u}(\mathbf{X})$	$\mathbf{u}(\mathbf{Y})$, $\mathbf{u}(\mathbf{Z})$
$S^{bb}(\mathbf{X}\|\mathbf{Y},\mathbf{Z})$	$\mathbf{B}(\mathbf{X})$	$\mathbf{B}(\mathbf{Y})$, $\mathbf{B}(\mathbf{Z})$
$S^{ub}(\mathbf{X}\|\mathbf{Y},\mathbf{Z})$	$\mathbf{u}(\mathbf{X})$	$\mathbf{B}(\mathbf{Y})$, $\mathbf{B}(\mathbf{Z})$
$S^{bu}(\mathbf{X}\|\mathbf{Y},\mathbf{Z})$	$\mathbf{B}(\mathbf{X})$	$\mathbf{u}(\mathbf{Y})$, $\mathbf{u}(\mathbf{Z})$

2.1. Combined energy transfers in MHD turbulence. The sums in Eqs. (13, 14) involve all the modes of the system. However, considerable details are seen when we focus on a pair of triads, $(\mathbf{X},\mathbf{Y},\mathbf{Z})$ and $(-\mathbf{X},-\mathbf{Y},-\mathbf{Z})$. Note that $\mathbf{X}+\mathbf{Y}+\mathbf{Z}=0$ for a triad. Under this condition, the above energy equations get reduced to the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_u(\mathbf{X}) = S^{uu}(\mathbf{X}|\mathbf{Y},\mathbf{Z}) + S^{ub}(\mathbf{X}|\mathbf{Y},\mathbf{Z}), \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_b(\mathbf{X}) = S^{bb}(\mathbf{X}|\mathbf{Y},\mathbf{Z}) + S^{bu}(\mathbf{X}|\mathbf{Y},\mathbf{Z}), \tag{16}$$

where

$$S^{uu}(\mathbf{X}|\mathbf{Y},\mathbf{Z}) = -\Im\left[\{\mathbf{X}\cdot\mathbf{u}(\mathbf{Z})\}\{\mathbf{u}(\mathbf{Y})\cdot\mathbf{u}(\mathbf{X})\} + \{\mathbf{X}\cdot\mathbf{u}(\mathbf{Y})\}\{\mathbf{u}(\mathbf{Z})\cdot\mathbf{u}(\mathbf{X})\}\right], \tag{17}$$

$$S^{bb}(\mathbf{X}|\mathbf{Y},\mathbf{Z}) = -\Im\left[\{\mathbf{X}\cdot\mathbf{u}(\mathbf{Z})\}\{\mathbf{B}(\mathbf{Y})\cdot\mathbf{B}(\mathbf{X})\} + \{\mathbf{X}\cdot\mathbf{u}(\mathbf{Y})\}\{\mathbf{B}(\mathbf{Z})\cdot\mathbf{B}(\mathbf{X})\}\right], \tag{18}$$

$$S^{ub}(\mathbf{X}|\mathbf{Y},\mathbf{Z}) = \Im\left[\{\mathbf{X}\cdot\mathbf{B}(\mathbf{Z})\}\{\mathbf{B}(\mathbf{Y})\cdot\mathbf{u}(\mathbf{X})\} + \{\mathbf{X}\cdot\mathbf{B}(\mathbf{Y})\}\{\mathbf{B}(\mathbf{Z})\cdot\mathbf{u}(\mathbf{X})\}\right], \tag{19}$$

$$S^{bu}(\mathbf{X}|\mathbf{Y},\mathbf{Z}) = \Im\left[\{\mathbf{X}\cdot\mathbf{B}(\mathbf{Z})\}\{\mathbf{u}(\mathbf{Y})\cdot\mathbf{B}(\mathbf{X})\} + \{\mathbf{X}\cdot\mathbf{B}(\mathbf{Y})\}\{\mathbf{u}(\mathbf{Z})\cdot\mathbf{B}(\mathbf{X})\}\right]. \tag{20}$$

These are the *combined energy transfers* to the wavenumber $\mathbf{X}$ from the other two wavenumbers $\mathbf{Y}$ and $\mathbf{Z}$ and they are summarized in Table 1. For these functions, the first argument is the receiver wavenumber, whereas the other two are the giver wavenumbers.

Fig. 1 illustrates some of the energy transfers of MHD transfers. From the structure of Eqs. (15)–(20), it is deduced that MHD turbulence has velocity-to-velocity ($U2U$), magnetic-to-magnetic ($B2B$), magnetic-to-velocity ($B2U$), and velocity-to-magnetic ($U2B$) energy transfers.

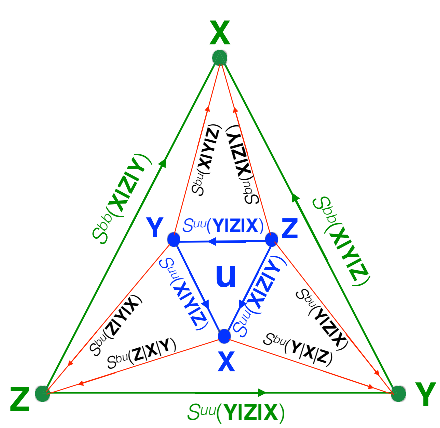

Fig. 1. A wavenumbers triad $(\mathbf{X},\mathbf{Y},\mathbf{Z})$ with $\mathbf{X}+\mathbf{Y}+\mathbf{Z}=0$ with blue and green symbols representing the velocity and magnetic modes, respectively. In the figure, S^{uu}, S^{bb}, S^{bu} are the mode-to-mode $U2U$, $B2B$, and $U2B$ transfers.

Table 2. Summary of various mode-to-mode energy transfers in MHD turbulence. Here the wavenumbers of the receiver, giver and mediator modes are $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, respectively.

ET	Receiver	Giver	Mediator	Formula
$S^{uu}(\mathbf{X}\vert\mathbf{Y}\vert\mathbf{Z})$	$\mathbf{u}(\mathbf{X})$	$\mathbf{u}(\mathbf{Y})$	$\mathbf{u}(\mathbf{Z})$	$-\Im\left[\{\mathbf{X}\cdot\mathbf{u}(\mathbf{Z})\}\{\mathbf{u}(\mathbf{Y})\cdot\mathbf{u}(\mathbf{X})\}\right]$
$S^{bb}(\mathbf{X}\vert\mathbf{Y}\vert\mathbf{Z})$	$\mathbf{B}(\mathbf{X})$	$\mathbf{B}(\mathbf{Y})$	$\mathbf{u}(\mathbf{Z})$	$-\Im\left[\{\mathbf{X}\cdot\mathbf{u}(\mathbf{Z})\}\{\mathbf{B}(\mathbf{Y})\cdot\mathbf{B}(\mathbf{X})\}\right]$
$S^{ub}(\mathbf{X}\vert\mathbf{Y}\vert\mathbf{Z})$	$\mathbf{u}(\mathbf{X})$	$\mathbf{B}(\mathbf{Y})$	$\mathbf{B}(\mathbf{Z})$	$\Im\left[\{\mathbf{X}\cdot\mathbf{B}(\mathbf{Z})\}\{\mathbf{B}(\mathbf{Y})\cdot\mathbf{u}(\mathbf{X})\}\right]$
$S^{bu}(\mathbf{X}\vert\mathbf{Y}\vert\mathbf{Z})$	$\mathbf{B}(\mathbf{X})$	$\mathbf{u}(\mathbf{Y})$	$\mathbf{B}(\mathbf{Z})$	$\Im\left[\{\mathbf{X}\cdot\mathbf{B}(\mathbf{Z})\}\{\mathbf{u}(\mathbf{Y})\cdot\mathbf{B}(\mathbf{X})\}\right]$

2.2. Mode-to-mode energy transfers. An important question is whether we can compute individual energy transfers to the wavenumber $\mathbf{X}$ from the wavenumber $\mathbf{Y}$ and to the wavenumber $\mathbf{X}$ from the wavenumber $\mathbf{Z}$. Dar *et al.* [2], Verma [3] and [4] derived formulas for these transfers. We will describe these formulas in the following discussion.

We denote the desired "mode-to-mode energy transfer" using $S^{fg}(\mathbf{X}|\mathbf{Y}|\mathbf{Z})$ that represents the energy transfer from the mode $\mathbf{g}(\mathbf{Y})$ to the mode $\mathbf{f}(\mathbf{X})$, with the mode $\mathbf{h}(\mathbf{Z})$ acting as a mediator. Hence, the receiver, giver and mediator fields are $\mathbf{f}, \mathbf{g}, \mathbf{h}$, respectively, and the corresponding wave numbers are, respectively, $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$. The receiver, giver and mediator wavenumbers appear as the arguments of S in the same order.

The energy transfer is a scalar and additive quality, similar to financial transactions. In addition, the energy exchanges must satisfy the following properties:

1. The sum of $S^{fg}(\mathbf{X}|\mathbf{Y}|\mathbf{Z})$ and $S^{fg}(\mathbf{X}|\mathbf{Z}|\mathbf{Y})$ is the combined energy transfer $S^{fg}(\mathbf{X}|\mathbf{Y},\mathbf{Z})$. That is,
$$S^{fg}(\mathbf{X}|\mathbf{Y}|\mathbf{Z}) + S^{fg}(\mathbf{X}|\mathbf{Z}|\mathbf{Y}) = S^{fg}(\mathbf{X}|\mathbf{Y},\mathbf{Z}). \tag{21}$$

2. The energy transfer from $\mathbf{g}(\mathbf{Y})$ to $\mathbf{f}(\mathbf{X})$, $S^{fg}(\mathbf{X}|\mathbf{Y}|\mathbf{Z})$, is equal and opposite to the energy transfer from $\mathbf{f}(\mathbf{X})$ to $\mathbf{g}(\mathbf{Y})$, $S^{gf}(\mathbf{Y}|\mathbf{X}|\mathbf{Z})$, i.e.
$$S^{fg}(\mathbf{X}|\mathbf{Y}|\mathbf{Z}) = -S^{gf}(\mathbf{Y}|\mathbf{X}|\mathbf{Z}). \tag{22}$$

In addition to the above properties, the mode-to-mode transfer functions must also satisfy Eqs. (15)–(20). It is easy to see that the following functions satisfy the above constraints [2, 3]:

$$\begin{aligned}
S^{uu}(\mathbf{X}|\mathbf{Y}|\mathbf{Z}) &= -\Im\left[\{\mathbf{X}\cdot\mathbf{u}(\mathbf{Z})\}\{\mathbf{u}(\mathbf{Y})\cdot\mathbf{u}(\mathbf{X})\}\right], && (23)\\
S^{bb}(\mathbf{X}|\mathbf{Y}|\mathbf{Z}) &= -\Im\left[\{\mathbf{X}\cdot\mathbf{u}(\mathbf{Z})\}\{\mathbf{B}(\mathbf{Y})\cdot\mathbf{B}(\mathbf{X})\}\right], && (24)\\
S^{ub}(\mathbf{X}|\mathbf{Y}|\mathbf{Z}) &= -\Im\left[\{\mathbf{X}\cdot\mathbf{B}(\mathbf{Z})\}\{\mathbf{B}(\mathbf{Y})\cdot\mathbf{u}(\mathbf{X})\}\right], && (25)\\
S^{bu}(\mathbf{X}|\mathbf{Y}|\mathbf{Z}) &= -\Im\left[\{\mathbf{X}\cdot\mathbf{B}(\mathbf{Z})\}\{\mathbf{u}(\mathbf{Y})\cdot\mathbf{B}(\mathbf{X})\}\right]. && (26)
\end{aligned}$$

They represent the velocity-to-velocity, magnetic-to-magnetic, magnetic-to-velocity, and velocity-to-magnetic energy transfers, respectively. The formulas in Eqs. (23)–(26) are the products of a scalar product of the giver and receiver modes (a) and of a scalar product of the mediator mode with the receiver wavenumber (b). These transfers are summarized in Table 2.

However, there is a complication. The above solution set basing on the terms of Eqs. (15)–(20) and Eqs. (21, 22) is not unique [2, 3]. We invoke the following physical arguments to show that the above functions are indeed the desired mode-to-mode energy transfers.

In Eq. (2), the nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{B}$ represents the advection of the magnetic field $\mathbf{B}$ by the velocity field $\mathbf{u}$. Therefore, for this process, $\mathbf{u}$ only mediates the $B2B$ energy transfer. In the equation for the magnetic energy, Eq. (12), the $B2B$ energy transfer is via the term $u_j\partial_j\left(\frac{1}{2}B_iB_i\right)$. Here, $\mathbf{u}$, which is to the left of the derivative operator, acts

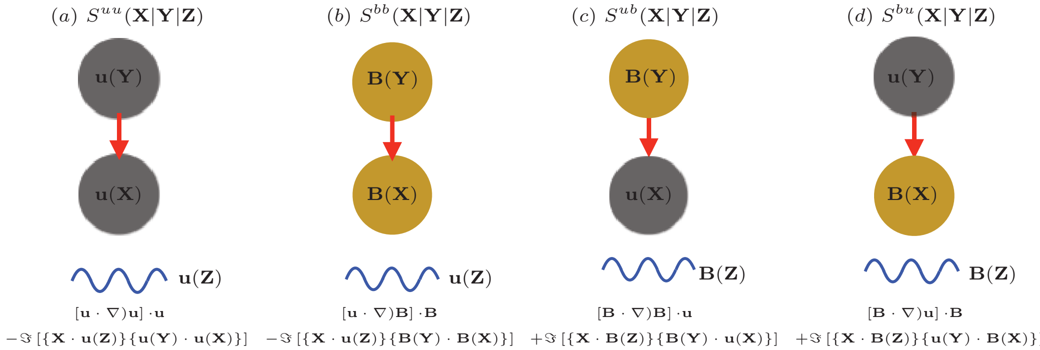

Fig. 2. Schematic diagram exhibiting mode-to-mode energy transfers in MHD turbulence: (*a*) *U2U*, (*b*) *B2B*, (*c*) *B2U*, and (*d*) *U2B*. Here the wavy lines represent the mediator modes.

as a mediator for the energy transfer between $\mathbf{B}$'s, which are to the right of the derivative operator. In the Fourier space, it corresponds to $S^{bb}(\mathbf{X}|\mathbf{Y}|\mathbf{Z})$ of Eq. (24) which is the energy transfer from $\mathbf{B}(\mathbf{Y})$ to $\mathbf{B}(\mathbf{X})$ with the mediation of $\mathbf{u}(\mathbf{Z})$ (wavy line) (see Fig. 2*b* for illustration).

Following the same arguments, it can be shown that Eq. (23) which is illustrated in Fig. 2*a* represents the $U2U$ energy transfer. When we relate it to the nonlinear term $u_j \partial_j \left(\frac{1}{2} u_i u_i\right)$ of Eq. (11), the mediator mode $\mathbf{u}(\mathbf{Z})$ corresponds to u_j (as in Eq. (24)), whereas the giver and receiver modes are $\mathbf{u}(\mathbf{Y})$ and $\mathbf{u}(\mathbf{X})$, respectively.

For the other two energy transfers of Eqs. (25, 26), the velocity field does not act as a advector. The magnetic field rather takes that role, as illustrated in Fig. 2*c*,*d*. For both these transfers, $\mathbf{B}(\mathbf{Z})$ acts as a mediator. In $S^{bu}(\mathbf{X}|\mathbf{Y}|\mathbf{Z})$, $\mathbf{B}(\mathbf{X})$ receives energy from $\mathbf{u}(\mathbf{Y})$, but in $S^{ub}(\mathbf{X}|\mathbf{Y}|\mathbf{Z})$, $\mathbf{u}(\mathbf{X})$ receives energy from $\mathbf{B}(\mathbf{Y})$.

Note that Verma [4] also provided symmetry-based arguments to show that Eqs. (23)–(26) are indeed the respective mode-to-mode energy transfers of MHD turbulence. These derivations, however, are beyond the scope of this paper.

3. Energy fluxes in MHD turbulence. The mode-to-mode energy transfers provide a perfect recipe for computing the energy fluxes for MHD turbulence. We consider a wavenumber sphere of the radius k_0. Various energy fluxes for this sphere are

$$\Pi^{u<}_{u>}(k_0) = \sum_{|\mathbf{Y}|\le k_0} \sum_{|\mathbf{X}|>k_0} S^{uu}(\mathbf{X}|\mathbf{Y}|\mathbf{Z}), \tag{27}$$

$$\Pi^{u<}_{b>}(k_0) = \sum_{|\mathbf{Y}|\le k_0} \sum_{|\mathbf{X}|>k_0} S^{bu}(\mathbf{X}|\mathbf{Y}|\mathbf{Z}), \tag{28}$$

$$\Pi^{b<}_{b>}(k_0) = \sum_{|\mathbf{Y}|\le k_0} \sum_{|\mathbf{X}|>k_0} S^{bb}(\mathbf{X}|\mathbf{Y}|\mathbf{Z}), \tag{29}$$

$$\Pi^{u>}_{b<}(k_0) = \sum_{|\mathbf{Y}|>k_0} \sum_{|\mathbf{X}|\le k_0} S^{ub}(\mathbf{X}|\mathbf{Y}|\mathbf{Z}), \tag{30}$$

$$\Pi^{u<}_{b<}(k_0) = \sum_{|\mathbf{Y}|\le k_0} \sum_{|\mathbf{X}|\le k_0} S^{ub}(\mathbf{X}|\mathbf{Y}|\mathbf{Z}), \tag{31}$$

$$\Pi^{u>}_{b>}(k_0) = \sum_{|\mathbf{Y}|>k_0} \sum_{|\mathbf{X}|>k_0} S^{bu}(\mathbf{X}|\mathbf{Y}|\mathbf{Z}). \tag{32}$$

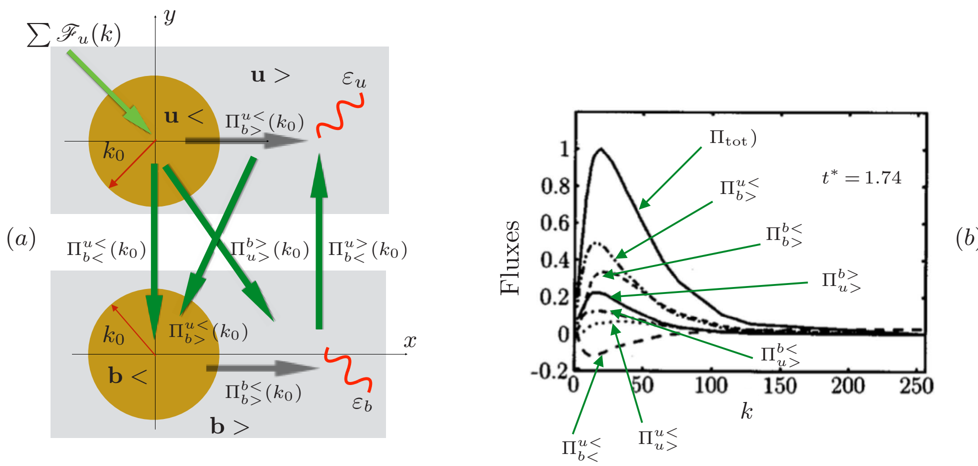

Fig. 3. (a) Various energy fluxes in MHD turbulence (see Sec. 3 for definitions). **u** $<$ and **u** $>$ represent, respectively, the velocity Fourier modes insides and outside the wavenumber sphere of a radius k_0. Similar notation for the magnetic field. $\sum \mathscr{F}_u(k)$ is the kinetic energy injection rate at small wavenumbers. $\Pi^{u<}_{b>}(k_0)$ is the energy flux arising due to the energy transfers from **u** $<$ to **b** $>$. (b) Energy fluxes computed for a decaying MHD turbulence simulation. Adopted from a figure of Debliquy *et al.* [5].

In the above formulas, the superscripts and subscripts represent the giver and receiver modes, respectively, whereas $<$ and $>$ represent the modes residing inside and outside the sphere, respectively. The above fluxes are illustrated in Fig. 3a.

Fig. 3b illustrates the energy fluxes computed by Debliquy *et al.* [5] using the numerical data of the decaying turbulence simulation. Some important observations from the plot are the following: (a) the $B2B$ energy transfer is positive; (b) there is an energy transfer from a large-scale magnetic field ($b <$) to a large-scale velocity field ($u <$); (c) the $U2U$ transfer is strongly suppressed. Note, however, that the energy transfers for forced MHD and from decaying MHD are quite different. For example, forced MHD always exhibits a positive $\Pi^{u<}_{b<}(k)$.

An important regime of MHD turbulence is the steady state which is obtained when the energy supply by $\mathbf{F}_{\text{ext}}$ matches with the total dissipation rate ϵ_{tot}. We denote the viscous dissipation rate of the velocity field by ϵ_u and the Joule dissipation rate of the magnetic field by ϵ_b (see Fig. 3a). Note that $\epsilon_{\text{tot}} = \epsilon_u + \epsilon_b$.

Here we present without proof some of the identities for a wavenumber sphere in the inertial range:

$$\Pi^{u<}_{u>}(k) + \Pi^{u<}_{b>}(k) + \Pi^{b<}_{b>}(k) + \Pi^{b<}_{u>}(k) = \Pi_{\text{tot}}(k) = \epsilon_{\text{tot}}, \tag{33}$$

$$\Pi^{\text{all}}_{u>}(k) = \Pi^{u<}_{u>}(k) + \Pi^{b<}_{u>}(k) + \Pi^{b>}_{u>}(k) = \epsilon_u, \tag{34}$$

$$\Pi^{\text{all}}_{b>}(k) = \Pi^{b<}_{b>}(k) + \Pi^{u<}_{b>}(k) + \Pi^{u>}_{b>}(k) = \epsilon_b, \tag{35}$$

$$\Pi^{u<}_{b<}(k) + \Pi^{u<}_{b>}(k) + \Pi^{u>}_{b<}(k) + \Pi^{u>}_{b>}(k) = \epsilon_b \tag{36}$$

$$\Pi^{u<}_{b<}(k) + \Pi^{u>}_{b<}(k) = \Pi^{b<}_{b>}(k), \tag{37}$$

$$\Pi^{u<}_{u>}(k) + \Pi^{u<}_{b<}(k) + \Pi^{u<}_{b>}(k) = \sum_{\mathbf{k}} \mathscr{F}_{\text{ext}}(\mathbf{k}) = \epsilon_{\text{tot}}. \tag{38}$$

Note that the above identities are not all independent.

In the next section, we describe how the energy fluxes can provide valuable inputs to the dynamo process which is the magnetic energy growth in astrophysical objects.

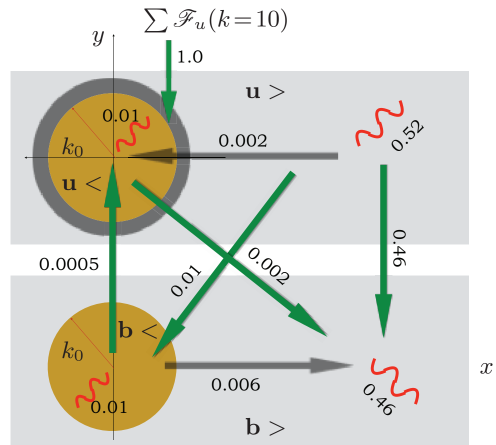

Fig. 4. Various energy fluxes for the nonhelical dynamo of Kumar and Verma [8]. These fluxes are of $t = 127$ eddy turnover time.

4. Application to large-scale dynamo. The dynamo process remains an unsolved problem [6]. Researchers have attempted to understand the dynamo process using different schemes: α-dynamo, experimental dynamos, dynamo transition [7], numerical dynamos, etc. Also, there are many parameters – Prandtl number, Reynolds number, geometry, forcing, etc. – that affect the dynamo process.

In this section, we present energy fluxes for large-scale dynamo that is forced at an intermediate scale. An important question is whether the large-scale magnetic field will grow. Kumar and Verma [8] studied such a dynamo in which the forcing was employed at a scale of 1/10 box size, i.e. $k \in (10, 11)$. They performed a numerical simulation of this dynamo and then analyzed various energy fluxes at different stages of the magnetic field growth. The flow was forced with a net kinetic energy injection rate of unity, but with a negligible kinetic and magnetic helicity. Thus, this is a non-helical dynamo. Also, the magnetic Prandtl number Pm was chosen to be unity.

Fig. 4 demonstrates various energy fluxes and dissipation rates towards the later stages ($t = 127$ eddy turnover time) of the dynamo process when the kinetic and magnetic energies reached a steady state. The central brown circle in Fig. 4 represents a sphere of radius $k_0 = 8$. Note that $k_0 < k_f$, so it represents large scales. When analyzing this figure, it was found that $\Pi^{u>}_{b<}(k_0) \approx 0.01$ was the most dominant energy input to the sphere. Also, the $b <$ sphere gives a relatively small amount of the energy to $u <$ and $b >$. The Joule dissipation in the sphere represented by a red wavy line appears to balance the energy input to the sphere. Thus, the large-scale magnetic field is sustained by the $\Pi^{u>}_{b<}$ energy flux. This energy flux is only 1% of the total energy injection rate, but this is sufficient because the Joule dissipation is quite weak due to the k^2 factor.

This dynamo has other interesting features, e.g, $\epsilon_u \approx 0.53$ (the sum of 0.52 and 0.01) and $\epsilon_b = 0.47$ at $t = 127$ time unit. Fig. 5 illustrates plots of the energy flux $\Pi^{u>}_{b<}$ at other time instances. These plots show that the large-scale magnetic field always gets a small amount of the energy from the forcing band.

Kumar *et al.* [9, 10] analyzed energy transfers for small and large magnetic Prandtl numbers and reported the dominant sources of inputs to the magnetic energy. These computations show that the energy fluxes provide valuable insights into the dynamo mechanism.

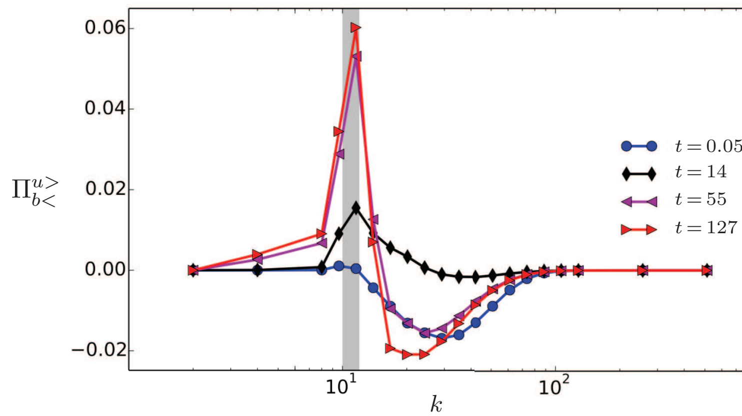

Fig. 5. Energy fluxes $\Pi^{u>}_{b<}$ in MHD turbulence at various stages of dynamo. All energy fluxes in Fig. 4 are for $t = 127$ eddy turnover time.

5. Conclusions. In the paper, we reported various energy transfers in MHD turbulence. In particular, the mode-to-mode, velocity-to-velocity, velocity-to-magnetic, magnetic-to-magnetic, and magnetic-to-velocity energy transfers were analyzed in detail. Then, we described six energy fluxes of MHD turbulence that provide valuable information on the energy transfers among large-scale and small-scale fields.

These energy fluxes and other related quantities are very useful for understanding turbulence dynamics. The paper illustrates how the energy fluxes help to understand the growth of the large-scale magnetic field when external forcing is employed at the intermediate scale. Helicity fluxes can be formulated in the same manner as energy fluxes. All these quantities are implemented in the TARANG code (open source software) for direct numerical simulation of MHD turbulence [11].

It was pointed out that the formalism of energy transfers is quite general, and it has been extended to other systems, such as passive scalar, thermal convection [12], shell model [13, 14], etc. It was also shown [2, 3] how the mode-to-mode energy transfer formalism removes the ambiguity in the shell-to-shell energy transfer computations encountered in the calculations that employ the formalism of the combined energy transfer.

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