Revisiting Reynolds and Nusselt numbers in turbulent thermal convection

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ABSTRACT

In this paper, we extend Grossmann and Lohse’s (GL) model [S. Grossmann and D. Lohse, “Thermal convection for large Prandtl numbers,” Phys. Rev. Lett. 86, 3316 (2001)] for the predictions of Reynolds number (Re) and Nusselt number (Nu) in turbulent Rayleigh–Bénard convection. Toward this objective, we use functional forms for the prefactors of the dissipation rates in the bulk and boundary layers. The functional forms arise due to inhibition of nonlinear interactions in the presence of walls and buoyancy compared to free turbulence, along with a deviation of the viscous boundary layer profile from Prandtl–Blasius theory. We perform 60 numerical runs on a three-dimensional unit box for a range of Rayleigh numbers (Ra) and Prandtl numbers (Pr) and determine the aforementioned functional forms using machine learning. The revised predictions are in better agreement with the past numerical and experimental results than those of the GL model, especially for extreme Prandtl numbers.

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I. INTRODUCTION

A classical problem in fluid dynamics is Rayleigh–Bénard convection (RBC), where a fluid is enclosed between two horizontal walls with the bottom wall kept at a higher temperature than the top wall. RBC serves as a paradigm for many types of convective flows occurring in nature and in engineering applications. RBC is primarily governed by two parameters: the Rayleigh number, Ra, which is the ratio of the buoyancy and the dissipative force, and the Prandtl number, Pr, which is the ratio of kinematic viscosity and thermal diffusivity of the fluid. In this paper, we derive a relation to predict two important quantities—the Nusselt number, Nu, and the Reynolds number, Re, which are respective measures of large-scale heat transport and velocity in turbulent RBC.

The dependence of Nu and Re on RBC’s governing parameters (Ra and Pr) has been extensively studied in the literature.1–5 Malkus4 proposed Nu ~ Ra1/3 based on marginal stability theory. For very large Ra called the ultimate regime, Kraichnan6 deduced Nu ~ \sqrt{RaPr}, Re ~ \sqrt{Ra/Pr} for Pr ≤ 0.15 and Nu ~ RaPr1/2, Re ~ \sqrt{Ra/Pr} for 0.15 < Pr ≤ 1, with logarithmic corrections. Subsequently, Castaing et al.6 also argued that Nu ~ Ra0.7 and Re ~ Ra0.3 based on the existence of a mixing zone in the central region of the RBC cell, where hot rising plumes meet the mildly warm fluid. Castaing et al.6 also deduced that Reω ~ Ra1/7, where Reω is the Reynolds number based on the frequency ω of torsional azimuthal oscillations of the large-scale wind in RBC. Later, Shraiman and Siggia7 derived that Nu ~ Ra1/7 Pr1/7 and Re ~ Ra3/7 Pr−5/7 (with logarithmic corrections) using the properties of boundary layers. They also derived exact relations between Nu and the viscous and thermal dissipation rates.

Many experiments and simulations of RBC have been performed to obtain the scaling of Nu and Re. These studies also revealed a power-law scaling of Nu and Re as Nu ~ RaαPrβ and Re ~ RaδPrγ. For the scaling of Nu, the exponent α ranges from 1/4 for Pr ≪ 1 to approximately 1/3 for Pr ≥ 18,29 and β from approximately zero for Pr ≥ 1 to 0.14 for Pr ≪ 1.30,31 Thus, Nu has a relatively weaker dependence on Pr. For the scaling of Re, the exponent γ was observed to be approximately 2/5 for Pr ≪ 1, 1/2 for Pr ~ 1, and 3/5 for Pr ∼ 1;30,11,13,18,22,24,32–35 and δ has been observed to decrease from 0.7 for Pr ≤ 1 to −0.95 for Pr ∼ 1.30,36 A careful examination of the results of the above references reveals that the above exponents also depend on the regime of Ra as well. The
The outline of the paper is as follows: In Sec. II, we discuss the governing equations of RBC and briefly explain the GL model. Then, we extend the GL framework by using functional forms for the prefactors of the dissipation rates in the bulk and boundary layers and incorporate the deviation in the scaling of viscous boundary layer thickness described earlier. Simulation details are provided in Sec. III. In Sec. IV, we report the scaling of boundary layer thicknesses and dissipation rates using our data, following which we describe the machine-learning tools used to determine the aforementioned functional forms. We also test the revised predictions with experiments and numerical simulations, and compare them with those of the GL model. We conclude in Sec. V.

II. RBC EQUATIONS AND THE GL MODEL

We consider RBC under the Boussinesq approximation, whose governing equations are as follows,

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p/\rho_0 + \alpha g T \hat{z} + \nu \nabla^2 \mathbf{u}, \\
\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T &= \kappa \nabla^2 T, \\
\nabla \cdot \mathbf{u} &= 0,
\end{align*}
\]

where \( \mathbf{u} \) and \( p \) are the velocity and pressure fields, respectively, \( T \) is the temperature field, \( \nu \) is the kinematic viscosity, \( \kappa \) is the thermal diffusivity, \( \alpha \) is the thermal expansion coefficient, \( \rho_0 \) is the mean density of the fluid, and \( g \) is the acceleration due to gravity.

Using \( d \) as the length scale, \( \sqrt{\alpha g \Delta d} \) as the velocity scale, and \( \Delta \) as the temperature scale, we non-dimensionalize Eqs. (1)–(3) that yield

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + T \hat{z} + \frac{\nu}{\text{Ra}} \nabla^2 \mathbf{u}, \\
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T &= \frac{1}{\sqrt{\alpha \text{Pr}}} \nabla^2 T, \\
\nabla \cdot \mathbf{u} &= 0,
\end{align*}
\]

where \( \text{Ra} = \alpha g \Delta d^3/(\nu \kappa) \) is the Rayleigh number and \( \text{Pr} = \nu/\kappa \) is the Prandtl number. The large-scale velocity and heat transfer are quantified by two important non-dimensional quantities, namely, the Reynolds number (Re) and the Nusselt number (Nu). The Nusselt number, Nu, is the ratio of the total heat flux to the conductive heat flux and is defined as \( \text{Nu} = \langle u_z \rangle / (\kappa \Delta/\delta) \). The Reynolds number Re is defined as \( \text{Re} = U d / \nu \), where \( U \) is the large-scale velocity. In our work, we will consider \( U \) to be the root mean square (rms) velocity, that is, \( U = \sqrt{\langle u_x^2 + u_y^2 + u_z^2 \rangle} \), where \( \langle \cdot \rangle \) represents the volume average.

The dissipation rates of kinetic and thermal energies, represented as \( \epsilon_k \) and \( \epsilon_T \), respectively, are important quantities in our study. These are defined as \( \epsilon_k = 2\nu \langle \nabla S \rangle / \delta \) and \( \epsilon_T = \kappa \langle |\nabla T|^2 \rangle / \delta \), where \( S_j \) is the strain rate tensor, Shraiman and Sigga argued that the scaling of Re and Nu deviates significantly from that of \( \epsilon_k \) and \( \epsilon_T \). The above studies show that the scaling of Re and Nu depends on the regime of Ra and Pr, highlighting the need for a unified model that encompasses all the regimes. Grossmann and Lohse constructed one such model, henceforth referred to as the GL model. To derive this model, Grossmann and Lohse substituted the bulk and boundary layer contributions of viscous and thermal dissipation rates in the exact relations of Shraiman and Sigga. The bulk and boundary layer contributions were written in terms of Re, Nu, Ra, and Pr using the properties of boundary layers (Prandtl–Blasius theory) and those of hydrodynamic and passive scalar turbulence in the bulk. Finally, using additional crossover functions, Grossmann and Lohse obtained a system of equations for Re and Nu in terms of Ra, Pr, and four coefficients that were determined using simulations. However, it does not capture large Pr convection very accurately and has been reported to under-predict the Reynolds number. Note that the scaling exponent for Re has a longer range (0.40–0.60) compared to that for Nu (0.25–0.33); hence, the predictions for Re are more sensitive to modeling parameters. Furthermore, the GL model is based on certain assumptions that are not valid for RBC. For example, the model assumes the viscous and the thermal dissipation rate in the bulk scale as \( U^2/\delta \) and \( U^4/\delta^2 \) (for \( \text{Pr} \leq 1 \)), respectively, as in passive scalar turbulence with open boundaries. Here, \( U \) is the large-scale velocity, and \( \Delta \) and \( \delta \) are, respectively, the temperature difference and distance between the top and bottom walls. However, subsequent studies of RBC have shown that the aforementioned viscous and thermal dissipation rates in the bulk are suppressed by approximately \( \text{Ra}^{-0.2} \) for \( \text{Pr} \sim 1 \). The above suppression is due to the inhibition of nonlinear interactions because of wall and buoyancy. Moreover, recent studies have revealed that the viscous boundary layer thickness in RBC considerably deviate from \( \text{Re}^{-1/2} \) as assumed in the GL model.

In the present work, we address the above limitations of the GL model and propose a new relation for the Reynolds and Nusselt numbers involving a cubic polynomial equation for Re and Nu. For implementation of the viscous and thermal dissipation rates in the bulk and boundary layers, we employ machine-learning tools on 60 datasets that were obtained using numerical simulations of RBC. The new relation rectifies some of the limitations of the GL model, especially for small and large Prandtl numbers.

The new relation rectifies some of the limitations of the GL model, especially for small and large Prandtl numbers.
The values of the constants, obtained from experiments, are $c_1 = 1.38$, $c_2 = 8.05$, $c_3 = 0.0252$, $c_4 = 0.487$, $a = 0.922$, and $Re_c = 3.401$. The above equations can be solved iteratively to obtain $Re$ and $Nu$ for given $Ra$ and $Pr$.

Although the GL model has been quite successful in predicting $Re$ and $Nu$, it has certain deficiencies due to some assumptions that are invalid for RBC. First, recent studies reveal that the relation $\delta_u \sim Re^{1/2}$ for the viscous boundary layers is not strictly valid for RBC.\textsuperscript{50-52} The viscous boundary layer thickness becomes a progressively weaker function of $Re$ as $Pr$ is increased.\textsuperscript{54} Thus, the relation given by Eq. (12) is not accurate. Second, as discussed earlier, studies have shown that for $Pr \sim 1$, the thermal and viscous dissipation rates in the bulk are suppressed relative to free turbulence.\textsuperscript{55-59}

Contrast the above relations with Eqs. (11) and (13)\textsuperscript{59,59} used in the GL model. This clearly signifies that $c_1$ and $c_4$ from Eqs. (11) and (13) cannot be treated as constants. Thus, it becomes imperative to study how $c_1$ varies with $Ra$ and $Pr$ in different regimes of RBC.

We propose a modified relation for $Re$ and $Nu$ by incorporating the aforementioned suppression of the total dissipation rates, as well as the modified law for the viscous boundary layers. Toward this objective, we make the following modifications to Eqs. (11)–(14),

\[
\frac{1}{V} \hat{D}_{b,\text{bulk}} \sim \frac{U^3}{d} \propto \Re^3, \quad \frac{1}{V} \hat{D}_{b,\text{BL}} \sim \frac{vU^2}{\delta_0^2} \propto \Re^{2.5}, \quad \frac{1}{V} \hat{D}_{T,\text{bulk}} \sim \frac{U^2}{\delta_0^2} \propto \Re^{2/3} \Pr \mathrm{Nu}^2, \quad \frac{1}{V} \hat{D}_{T,\text{BL}} \sim \frac{\kappa \Delta \delta_T}{\delta_0^2} \propto \Re^{2/3} \Pr \mathrm{Nu}^{1/2},
\]

\[
\frac{1}{V} \hat{D}_{T,\text{bulk}} \sim \frac{\kappa \Delta \delta_T}{\delta_0^2} \propto \Re^{2/3} \Pr \mathrm{Nu}^{1/2}.
\]

Note that we replaced the coefficients $c_i$ with functions $f_i(Ra, Pr)$. Furthermore, we do not express $d/\delta_0$ in terms of $Re$ in Eq. (19). The above modified formulas are inserted in the exact relations of Shraiman and Sigga\textsuperscript{12} that lead to

\[
Nu = f_1(Ra, Pr)\Re^{3/2} + f_2(Ra, Pr)\frac{d}{\delta_0} \Re^2,
\]

\[
Nu = f_3(Ra, Pr)\Re^{3/2} + 2f_4(Ra, Pr)\frac{d}{\delta_0} \Re^2
\]

The functions $f_i(Ra, Pr)$ will be later determined using our simulation results. For the sake of brevity, we will skip the arguments within the parenthesis of $f_i$'s henceforth.

Equations (22) and (23) constitute a system of two equations with two unknowns (Re and Nu). To solve these equations, we will now reduce them to a cubic polynomial equation for Re by eliminating Nu. We rearrange Eq. (23) to obtain

\[
Nu = \frac{f_3}{1 - 2f_4} \Re \Pr.
\]
Substitution of Eq. (24) in Eq. (22) yields the following cubic equation for Re:

\[ f_2 \frac{d}{\delta_u} Re^2 - \frac{f_3}{1 - 2f_4} \frac{Ra}{Pr} Re + \frac{Ra}{Pr^2} = 0. \]  

(25)

The above equation for Re can be solved for a given Ra and Pr once \( f_1 \) and \( \delta_u \) have been determined. We determine Nu using Eq. (24) once Re has been computed.

Now, we will show that in the limit of the viscous dissipation rate dominating in the bulk or in the boundary layers (\( \delta_u \text{bulk} \gg \delta_u \text{BL} \) or vice versa), Eqs. (22) and (23) are reduced to power-law expressions for Re and Nu. In the following discussion, we consider scaling for these limiting cases.

Case 1: \( \delta_u \text{bulk} \gg \delta_u \text{BL} \).

First, let us consider the case where the viscous dissipation rate in the bulk is dominant. This regime is expected for large Ra (\( \gg 10^6 \)) or for small Pr (\( \ll 1 \)), where the boundary layers are thin. In this regime, \( f_2 (d/\delta_u) Re^2 \ll f_1 Re^3 \). Assuming \( Nu \gg 1 \), Eq. (22) reduces to

\[ Nu \approx \frac{Ra}{Pr^2} = f_1 Re^3. \]  

(26)

Using Eqs. (24) and (26), we arrive at

\[ Re = \sqrt{\frac{f_3}{f_1 (1 - 2f_4)} \frac{Ra}{Pr}} \]  

(27)

\[ Nu = \sqrt{\frac{1}{f_3} \left( \frac{f_3}{1 - 2f_4} \right)^3 \frac{RaPr}{3}.} \]  

(28)

Note that \( f_1 \) and \( f_3 \) are expected to be constants and \( f_2 = 0 \) when the boundary layers are absent (as in a periodic box) or weak (as in the ultimate regime proposed by Kraichnan). For this case, \( Re \sim \sqrt{Ra/Pr} \) and \( Nu \sim \sqrt{RaPr} \), consistent with the arguments of Kraichnan for large Ra and small Pr. However, for RBC with walls, the relations for Re and Nu will deviate from the above relations because \( f_1 \) and \( f_3 \) are functions of Ra and Pr.

Case 2: \( \delta_u \text{BL} \gg \delta_u \text{bulk} \).

Now, we consider the other extreme when the viscous dissipation rates in the boundary layers are dominant, which is expected for small Ra (\( \ll 10^6 \)) or for large Pr (\( \gg 7 \)). In this regime, again assuming \( Nu \gg 1 \), Eq. (22) reduces to

\[ Nu \approx \frac{Ra}{Pr^2} = f_2 \frac{d}{\delta_u} Re^2. \]  

(29)

Using Eqs. (24) and (29), we obtain

\[ Re = \left\{ \frac{f_3}{f_1 (1 - 2f_4)} \frac{\delta_u}{d} \right\} \frac{Ra}{Pr}, \]  

(30)

\[ Nu = \left\{ \frac{1}{f_3} \left( \frac{f_3}{1 - 2f_4} \right)^2 \right\} \frac{Ra}{Pr}. \]  

(31)

We will examine these cases once we deduce the forms of \( f_i \) using our numerical simulations.

We remark that the aspect ratio of the RBC cell also influences the scaling of Ra and Pr. In the current work, we do not consider the effect of aspect ratio. We intend to include the aspect ratio dependence in a future work.

In Sec. III, we will discuss the simulation method.

III. SIMULATION DETAILS

We perform direct numerical simulations of RBC by solving Eqs. (4)–(6) in a cubical box of unit dimensions using the finite difference code SARAS.64,67 We carry out 60 runs for Pr ranging from 0.02 to 100 and Ra ranging from 5 \( \times 10^7 \) to \( 5 \times 10^8 \). The grid size was varied from 257\(^3\) to 1025\(^3\) depending on parameters. Refer to Tables I and II for the simulation details.

We impose isothermal boundary conditions on the horizontal walls and adiabatic boundary conditions on the sidewalls. No-slip boundary conditions were imposed on all the walls. A second-order Crank–Nicholson scheme was used for time-advancement, with the maximum Courant number kept at 0.2. The solver uses a multigrid method for solving the pressure-Poisson equations. We ensure a minimum of five points in the viscous and thermal boundary layers (see Tables I and II); this satisfies the resolution criterion of Grötzschbach and Verzicco and Camussi.77 The simulations are run up to 3–263 non-dimensional time units (\( \tau_{ND} \)) after attaining a steady state. For post-processing, we employ a central difference method for spatial differentiation and Simpson’s method for computing the volume average.

In order to resolve the smallest scales of the flow, we ensure that the grid spacing \( \Delta x \) is smaller than the Kolmogorov length scale \( \eta = (\nu^3 \epsilon_x^{-2})^{1/4} \) for \( \nu \leq 1 \) and the Batchelor length scale \( \eta_T = (\nu \epsilon_T^{-1})^{1/4} \) for \( \nu > 1 \). We numerically compute \( \epsilon_x \) and \( \epsilon_T \) and use these values to compute \( Nu_x \) and \( Nu_T \) employing Shraiman and Sigga’s exact relations66,67 [see Eqs. (7) and (8)]. The Nusselt numbers computed using \( (u, T) \) match with \( Nu_x \) and \( Nu_T \) within two percent on an average; this further confirms that our runs are well-resolved (see Tables I and II). All the above quantities are averaged over 12 to 259 snapshots taken at equal time intervals after attaining a steady state.

In Sec. IV, we analyze our numerical results, construct the cubic polynomial relation for Re and Nu using the data from our simulations, and compare our revised predictions with those of the GL model.

IV. RESULTS

Using our numerical data, we determine the scaling of dissipation rates, boundary layer thicknesses, and the functional forms of \( f_i \). We construct the relations for Re and Nu given by Eqs. (22) and (23) using these inputs and compare the revised predictions with those of the original GL model. We also analyze how the proposed relation performs in the limit of \( \delta_u \text{bulk} \gg \delta_u \text{BL} \) and vice versa.

A. Viscous and thermal dissipation rates

Here, we examine the scaling of viscous and thermal dissipation rates and explore how their scaling deviates from that of free turbulence. First, we present theoretical arguments on the above scaling, following which we verify our arguments with our numerical results.
Recall from Sec. I that the Reynolds number scales as $\text{Re} \sim \text{Ra}^{0.2}$ for $\text{Pr} \sim 1$ and $\text{Re} \sim \text{Ra}^{0.5}$ for $\text{Pr} \gg 1$, and the Nusselt number scales as $\text{Nu} \sim \text{Ra}^{0.3}$ for $\text{Pr} \gtrsim 1$. Substituting the above relations in Eqs. (33) and (34) yields

$$e_u \sim \frac{U^3}{d}, \quad e_T \sim \frac{U \Delta^2}{d}.$$  \hspace{1cm} (32)

However, in wall-bounded convection, the scaling of the dissipation rates is different. To understand this, let us rewrite the exact relations of Shraiman and Siggia given by Eqs. (7) and (8) as

$$e_u = \frac{U^3}{d} \frac{1}{\text{Re}} \left( \text{Nu} - 1 \right) \frac{\text{Ra}}{\text{Pr}}, \quad e_T = \frac{U \Delta^2}{d} \frac{1}{\text{RePr}} \text{Nu}.$$  \hspace{1cm} (33)

Recall from Sec. I that the Reynolds number scales as $\text{Re} \sim \text{Ra}^{1/2}$ for $\text{Pr} \sim 1$ and $\text{Re} \sim \text{Ra}^{0.5}$ for $\text{Pr} \gg 1$, and the Nusselt number scales as $\text{Nu} \sim \text{Ra}^{0.3}$ for $\text{Pr} \gtrsim 1$. Substituting the above relations in Eqs. (33) and (34) yields

$$e_u \sim \left\{ \begin{array}{ll} \frac{U^3}{d} \frac{\text{Ra}^{-0.2}}{\text{Pr}}, & \text{Pr} \sim 1, \\ \frac{U^3}{d} \frac{\text{Ra}^{-0.5}}{\text{Pr}^{1/2}}, & \text{Pr} \gg 1, \end{array} \right.$$  \hspace{1cm} (35)

instead of $U^3/d$, and

$$e_T \sim \left\{ \begin{array}{ll} \frac{U \Delta^2}{d} \frac{\text{Ra}^{-0.2}}{\text{Pr}}, & \text{Pr} \sim 1, \\ \frac{U \Delta^2}{d} \frac{\text{Ra}^{-0.3}}{\text{Pr}^{1/2}}, & \text{Pr} \gg 1, \end{array} \right.$$  \hspace{1cm} (36)

instead of $U \Delta^2/d$. Pandey and Verma \cite{Pandey22} and Pandey et al. \cite{Pandey23} argued that the additional Ra dependence is due to the suppression of nonlinear interactions due to the presence of walls. Some Fourier
modes that are otherwise present in free turbulence are absent in wall-bounded RBC; this results in several channels of nonlinear interactions and energy cascades to be blocked.\textsuperscript{5} Note that the horizontal walls seem to have a more pronounced effect on the aforementioned suppression than the lateral walls, as Schmidt et al.\textsuperscript{31} observed passive scalar scaling for homogeneous laterally confined RBC. In addition, buoyancy also appears to suppress the energy cascade rate,\textsuperscript{49} similar to the role played by the magnetic field in magnetohydrodynamic turbulence.\textsuperscript{60}

Now, for Pr $\ll 1$, recall that Re $\sim Ra^{0.42}$ and Nu $\sim Ra^{0.25}$ (see Sec. 1). Substitution of these expressions in Eqs. (33) and (34) yields

$$\epsilon_u \sim \frac{U^3}{d}, \quad \epsilon_T \sim \frac{UA^2}{d} Ra^{-0.17}.$$  \hspace{1cm} (37)

Thus, the viscous dissipation rate scales similar to free turbulence for small Pr. However, the additional Ra dependence is still present in the scaling of thermal dissipation rates because of the presence of thick thermal boundary layers.

Using our data, we numerically compute the viscous and thermal dissipation rates and normalize them with $U^3/d$ and $UA^2/d$, respectively. We plot the normalized dissipation rates vs Ra and exhibit these plots in Figs. 1(a) and 1(b). We observe that for small Pr, the normalized viscous dissipation rate is independent of Ra, whereas for larger Pr, the aforementioned quantity decreases with Ra. The decrease becomes steeper as Pr increases, with $\epsilon_u/(U^3/d) \sim Ra^{-0.21}$ for Pr = 1 and $\sim Ra^{-0.45}$ for Pr = 100. The normalized thermal dissipation rate decreases with Ra for all Pr values, with $\epsilon_T/(UA^2/d) \sim Ra^{-0.15}$ for Pr = 0.02 to $\sim Ra^{-0.28}$ for Pr = 100, which are consistent with the earlier estimates.
In Subsection IV B, we discuss the computations of the boundary layer thicknesses and their dependence on Re and Nu for different Pr.

B. Boundary layer thicknesses

There are several ways to define the viscous and thermal boundary layer thicknesses in RBC. In our paper, the viscous boundary layer thickness $\delta_u$ is defined as the depth where a linear fit of the velocity profile near the wall intersects with the tangent to the velocity profile at its local maximum. Similarly, the thermal boundary layer thickness $\delta_T$ is defined as the depth where a linear fit of the temperature profile near the wall intersects with the mean temperature $T = 0.5$. The above methods are described in detail in Refs. 1, 61, and 64.

Using the data generated from our simulations, we first compute the thicknesses of the thermal and viscous boundary layers. We report the average thicknesses of the viscous boundary layers near all the six walls and the thermal boundary layers near the top and bottom walls. We examine the validity of the Prandtl–Blasius relation of $\delta_u \sim \text{Re}^{-0.3}$ for the viscous boundary layers and $\delta_T = 0.5 \text{Nu}^{-1}$ for the thermal boundary layers. Toward this objective, we plot $\delta_T \text{Nu}$ vs Nu in Fig. 2(a) and $\delta_u \text{Re}^{1/2}$ vs Re in Fig. 2(b).

We observe from Fig. 2(a) that $\delta_T \text{Nu} = 1/2$, independent of Nu, which is consistent with the definition. On the other hand, from Fig. 2(b), it is evident that $\delta_u \text{Re}^{1/2}$ is constant in Re only for Pr = 0.5 and 0.1. However, $\delta_u \text{Re}^{1/2}$ increases as $\sim \text{Re}^{-0.31}$ for large Pr and decreases marginally as $\sim \text{Re}^{-0.07}$ for Pr = 0.02. This shows that for large Pr, $\delta_u$ becomes a weak function of Re; this is consistent with the observation of Breuer et al. We also plot Grossmann and Lohse’s estimate of viscous boundary layer thickness, which is given by $g(\sqrt{\text{Re}_{\text{c}}/\text{Re}})$, here, $g(x) = x(1 + x^2)^{-3/4}$ and $\text{Re}_{\text{c}} = 3.401$. It is clear that Grossmann and Lohse’s estimate deviates significantly from the actual values.

Therefore, we cannot assume $\delta_u \sim g(\text{Re}^{-1/2})$ for viscous boundary layers in RBC, and it is more prudent to obtain the scaling of $f_i \delta_u^{1/4}$ with Ra, where $f_i$ is the function from Eq. (19). The above deviation from the Prandtl–Blasius profile has also been observed in previous studies. This is because $\delta_u \sim \text{Re}^{-1/2}$ is valid asymptotically for very large Reynolds numbers.

C. $f_i$ vs Ra for different Pr

In this subsection, we numerically compute $f_i$ using our simulation data and discuss how these quantities vary with Ra for different Pr. We also obtain the limiting cases for the scaling of $f_i$ with Ra.

We numerically compute the total viscous and thermal dissipation rates in the bulk and in the boundary layers for all the simulation runs. Using these values and boundary layer thicknesses, we compute $f_1$, $f_2$, $f_3$, and $f_4$ and plot them vs Ra in Fig. 3. We observe that $f_1$ and $f_4$ are, in general, not constants as in free turbulence. $f_3$ decreases with Ra except for Pr = 0.1 and 0.02, where it is nearly constant. The above decrease is more prominent for large Pr ($\geq 50$), where $f_3 \sim \text{Ra}^{-0.33}$. In a similar fashion, $f_1$ also decreases with Ra for...
FIG. 3. Plots of (a) $f_1$, (b) $f_2$, (c) $f_3$, and (d) $f_4$ vs $Ra$. The error bars represent the standard deviation of the dataset with respect to the temporal average. $f_2$ remains roughly independent of $Ra$ and $Pr$, albeit with fluctuations; however, $f_1$, $f_3$, and $f_4$ decrease with $Ra$.

All Pr values and is more pronounced for large Pr ($f_3 \sim Ra^{-0.26}$) and less pronounced for small Pr ($f_3 \sim Ra^{-0.15}$). The above observations imply that the scaling of the dissipation rates in the bulk is similar to that in the entire volume\cite{58,59} (see Sec. IV A). This is because the bulk occupies a large fraction of the total volume, and its contribution to the total dissipation is significant.\cite{58,59}

The $Ra$ and $Pr$ dependences of $f_2$ cannot be clearly established from Fig. 3(b); we can only infer that $f_2$ is independent of $Ra$ and $Pr$, albeit with significant fluctuations. This is consistent with $\epsilon_{u,bl} \sim vU^2/\delta_u$ as predicted by Grossmann and Lohse.\cite{49,50} The function $f_2$ of Fig. 3(d) appears flat, but a careful examination shows that $f_2$ decreases weakly with $Ra$, with $f_2 \sim Ra^{-0.013}$ for small $Pr$ and $f_2 \sim Ra^{-0.0036}$ for large $Pr$. The reason for the marginal decrease of $f_2$ with $Ra$ needs investigation and is not in the scope of this paper.

As discussed earlier, the solution of Eq. (25) for $Re$ and $Nu$ depends on the quantity $f_2 \delta_u^{-1}$. Hence, we plot this quantity vs $Ra$ for different $Pr$ in Fig. 4. Since $f_2$ is nearly constant, $f_2 \delta_u^{-1}$ is inversely proportional to the viscous boundary layer thickness. Thus, $f_2 \delta_u^{-1}$ increases marginally for large $Pr$ ($\sim Ra^{0.052}$) and steeply for small $Pr$ ($\sim Ra^{-0.26}$), which is in agreement with the scaling of viscous boundary layer thickness discussed in Sec. IV B.

In Subsection IV D, we describe the machine-learning tools used to determine the functional forms of $f_i$.

D. Machine-learning algorithm to obtain $f_i(Ra, Pr)$

So far, we have examined the variation of $f_i$ with only $Ra$ for different Prandtl numbers and obtained the limiting cases. Now, using machine-learning and matching functions, we will combine these scalings to determine $f_i$ as functions of both $Ra$ and $Pr$. We make use of the machine-learning software WEKA\cite{70} for obtaining the functional forms of $f_i$. The values of $f_i$ computed for every $Ra$ and $Pr$ using our simulation data serve as training sets for our machine-learning algorithm. For simplicity, we will look for a power-law relation of the form $f_i = A Ra^\alpha Pr^\beta$, take logarithms of this expression, and employ linear regression to obtain $A$, $\alpha$, and $\beta$. The linear regression algorithm works by estimating coefficients for a hyperplane that best fits the training data using the least squares method.

FIG. 4. Plot of $f_2 \delta_u^{-1}$ vs $Ra$. The error bars represent the standard deviation of the dataset with respect to the temporal average. The dependence of $f_2 \delta_u^{-1}$ on $Ra$ is stronger for small $Pr$ and becomes weaker as $Pr$ increases.
Since the dependence of $f_i$ on $Ra$ is not uniform (see Sec. IV C), we split our parameter space into three regimes such that for each regime, the scaling of $f_i$ with $Ra$ is approximately the same. We choose the regimes as follows:

- **Small Pr**: $Pr \leq 0.5$,
- **Moderate Pr**: $0.5 \leq Pr \leq 6.8$,
- **Large Pr**: $Pr \geq 6.8$.

We then determine the prefactor $A$ and the exponents $\alpha$ and $\beta$ for each regime. To ensure continuity between the regimes, we introduce the following matching functions:

$$H_1(Pr) = \frac{1}{1 + e^{k_1(0.5-Pr)}}$$

$$H_2(Pr) = \frac{1}{1 + e^{k_2(Pr-0.5)}} - \frac{1}{1 + e^{k_2(Pr-6.8)}}$$

$$H_3(Pr) = \frac{1}{1 + e^{k_2(Pr-6.8)}}$$

where $k_1$ and $k_2$ are taken to be 10 and 0.75, respectively. The functions $H_1$, $H_2$, and $H_3$ become unity inside the regimes given by $Pr < 0.5$, $0.5 < Pr < 6.8$, and $Pr > 6.8$, respectively, and become negligible outside their regimes. The value of these functions is 1/2 in the boundaries of their respective regimes. See Fig. 5 for an illustration of the behavior of the matching functions. Using these functions and employing regression for each regime, we obtain the following fits for $f_i$:

$$f_1 = 0.67H_1Pr^{0.28} + 27H_2Ra^{-0.21}Pr^{0.55} + 170H_3Ra^{-0.34}Pr^{0.78},$$

$$f_2^2 = 4.4H_1Ra^{0.29}Pr^{-0.26} + 7.4H_2Ra^{0.22}Pr^{-0.29} + 27H_3Ra^{0.14}Pr^{-0.18},$$

$$f_3 = 0.095H_1Ra^{-0.15}Pr^{-0.17} + 0.25H_2Ra^{-0.21}Pr^{-0.17} + 0.45H_3Ra^{-0.25}Pr^{-0.093},$$

$$f_4 = 0.46H_1Ra^{-0.013}Pr^{0.10} + 0.43H_2Ra^{-0.0081}Pr^{0.053} + 0.39H_3Ra^{-0.0036}Pr^{0.093}.$$  

The average deviation between the $f_i$’s predicted by the fits and the actual values are 24%, 19%, 12%, and 58% for $f_1$, $f_2$, $f_3$, and $f_4$, respectively. As we will see later, incorporation of the aforementioned functional forms results in more accurate predictions than the GL model; thus, the above uncertainty in $f_i$ is acceptable. In the Appendix, we employ the same regression algorithm over a reduced training set consisting of half of our data points and show that the fits so obtained are close to Eqs. (41)–(44). Thus, the estimated parameter values are reasonably robust.

Having obtained the functional forms of $f_i$, we can plug them in Eqs. (25) and (24) to complete the relation for $Re$ and $Nu$. We remark that $f_i$ obtained above are valid for RBC cells with a unit aspect ratio. We suspect that they are weak functions of the aspect ratio; this study will be taken up in the future work. Furthermore, efforts are ongoing to make the functional forms of $f_i(Ra, Pr)$ more compact.

### E. Enhancement of the GL model

In this subsection, we will examine the enhancement of the GL model brought about by using the obtained functional forms for the prefactors of the dissipation rates. We will test both, the original GL model and the revised estimates with our numerical results, as well as those of Scheel and Schumacher (Pr = 0.005 and 0.02), Wagner and Shishkina (Pr = 0.7), Emran and Schumacher (Pr = 0.7), Kaczorowski and Xia (Pr = 4.38), and Horn, Shishkina, and Wagner (Pr = 2547.9). We also include the experimental results of Cioni, Ciliberto, and Sommeria (Pr = 0.02), and Niemela et al. (Pr = 0.7) for our comparisons. The simulations of Wagner and Shishkina and Kaczorowski and Xia involved a cubical cell such as ours, whereas the rest of the above simulations and experiments involved a cylindrical cell. All the above work involve RBC cells with a unit aspect ratio. We compute the percentage deviations ($D_{Re}$ and $D_{Nu}$) between the estimated and actual values according to the following formula:

$$D = \left(\frac{\text{Actual value} - \text{Predicted value}}{\text{Actual value}}\right) \times 100.$$  

In Table III, we list the average of the deviations computed for all the points for every Pr.

In Figs. 6(a) and 6(b), we plot the normalized Reynolds number, $ReRa^{-0.5}$, computed using our simulation data and those of Refs. 10, 11, 16-18, and 32, vs $Ra$. To avoid clutter, we exhibit the results for $Pr < 1$ in Fig. 6(a) and those for $Pr \geq 1$ in Fig. 6(b). The dashed and solid curves in Fig. 6 denote $Re$ predicted by the GL model and our revised estimates, respectively. From Fig. 6 and Table III, it is clear that the revised estimates of $Re$ are in better agreement with the observed results compared to the original GL model, especially for extreme Prandtl numbers. Furthermore, the trend of estimated $Re$ is also in better agreement with the numerical and experimental
TABLE III. Quantitative comparison between the predictions of the GL model and the revised estimates of Nu and Re for different sets of simulation and experimental data. $D_{Re}$ is the percentage difference between the observed and predicted values of Re, and $D_{Nu}$ is the percentage difference between the observed and predicted values of Nu [see Eq. (45)]. Note that no data on Re are available for $Pr = 4.38$.

<table>
<thead>
<tr>
<th>Pr</th>
<th>Range of Ra (Re) (Revised estimate) (%)</th>
<th>$D_{Re}$ (GL Model) (%)</th>
<th>Range of Ra (Nu) (Revised estimate) (%)</th>
<th>$D_{Nu}$ (GL Model) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>$3 \times 10^5$–$10^7$</td>
<td>11</td>
<td>$3 \times 10^5$–$10^7$</td>
<td>9.6</td>
</tr>
<tr>
<td>0.02</td>
<td>$3 \times 10^5$–$3 \times 10^9$</td>
<td>9.1</td>
<td>$3 \times 10^5$–$3 \times 10^9$</td>
<td>10</td>
</tr>
<tr>
<td>0.1</td>
<td>$5 \times 10^2$–$10^4$</td>
<td>1.3</td>
<td>$5 \times 10^2$–$10^4$</td>
<td>3.1</td>
</tr>
<tr>
<td>0.5</td>
<td>$10^5$–$10^8$</td>
<td>1.9</td>
<td>$10^5$–$10^8$</td>
<td>1.4</td>
</tr>
<tr>
<td>0.7</td>
<td>$10^5$–$10^8$</td>
<td>6.8</td>
<td>$10^5$–$10^8$</td>
<td>3.8</td>
</tr>
<tr>
<td>1.0</td>
<td>$10^6$–$2 \times 10^9$</td>
<td>2.8</td>
<td>$10^6$–$2 \times 10^9$</td>
<td>3.6</td>
</tr>
<tr>
<td>4.38</td>
<td>...</td>
<td>...</td>
<td>$10^6$–$3 \times 10^9$</td>
<td>5.7</td>
</tr>
<tr>
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<td>$10^6$–$5 \times 10^9$</td>
<td>3.4</td>
<td>$10^6$–$5 \times 10^9$</td>
<td>6.5</td>
</tr>
<tr>
<td>50</td>
<td>$10^6$–$10^9$</td>
<td>6.0</td>
<td>$10^6$–$10^9$</td>
<td>3.2</td>
</tr>
<tr>
<td>100</td>
<td>$10^6$–$5 \times 10^9$</td>
<td>3.4</td>
<td>$10^6$–$5 \times 10^9$</td>
<td>2.7</td>
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<td>$10^5$–$10^9$</td>
<td>85</td>
<td>$10^5$–$10^9$</td>
<td>2.3</td>
</tr>
</tbody>
</table>

results (note that the trend of Re computed based on different large-scale velocities does not change even though there may be minor differences in absolute values). This improvement in the estimation of Re is crucial because the predictions of Re are more sensitive to modeling parameters compared to Nu due to a larger range of the scaling exponent.

In Figs. 7(a) and 7(b), we plot the normalized Nusselt number, $NuGr^{-0.3}$, computed using our simulation data along with those of Refs. 10, 11, and 16–19, vs Ra. We employ the Grashof number $Gr = Ra/Pr$ in the $y$ axis to avoid clutter; this is because $Nu \sim Ra^{0.3}$ (with a weak dependence on $Pr$). These figures, along with Table III, indicate that the revised estimates of Nu (solid curves) are more accurate compared to those predicted by the original GL model (dashed curves). It is interesting to note that for extreme conditions, the predictions of the GL model are even less accurate than those of Refs. 10, 11, and 16–19, even for $Pr > 1$. The error bars (shown only for our datasets) represent the standard deviation of the dataset with respect to the temporal average.
Prandtl numbers \((\text{Pr} = 0.005, 2547.9)\), the accuracy of the revised estimates of \(\text{Nu}\) is significantly improved with only 2.3% deviation from the actual values for \(\text{Pr} = 2547.9\%\) and 9.6% deviation for \(\text{Pr} = 0.005\). Contrast this with the GL model, where we observe 17% deviation for both \(\text{Pr} = 2547.9\) and 0.005. For \(\text{Pr} \sim 1\), the accuracy of the revised estimates of \(\text{Nu}\) and those predicted by the GL model are comparable, with the former being more accurate for \(\text{Ra} < 10^5\) but marginally less for larger \(\text{Ra}\). Thus, we observe an overall improvement in the predictions of \(\text{Nu}\), though it is not as significant as it was for \(\text{Re}\).

In Figs. 8(a) and 8(b), we contrast the \(\text{Pr}\) dependence on our estimates of \(\text{Re}\) and \(\text{Nu}\) and those of the GL model. Here, we plot the predictions of \(\text{Re}(\text{Pr})\) and \(\text{Nu}(\text{Pr})\) along with the actual values computed using our data and those of Refs. 11 and 16–19. We choose four Rayleigh numbers for our comparisons: \(10^4\), \(10^5\), \(10^7\), and \(10^9\). As expected based on our earlier discussions, the revised estimates of \(\text{Re}(\text{Pr})\) are more accurate than those of the GL model [see Fig. 8(a)]. We also observe improvements in the predictions of \(\text{Nu}\), especially for \(\text{Pr} \ll 1\) and \(\text{Pr} \gg 1\) [see Fig. 8(b)]. This is again consistent with our earlier discussions.

The improvements, thus, in the predictions of \(\text{Re}\) and \(\text{Pr}\) underscore the importance of considering the additional \(\text{Ra}\) and \(\text{Pr}\) dependences on the scaling of the dissipation rates and the viscous boundary layers in convection.

### F. Limiting cases: Power-law expressions

Recall from Sec. II that Eqs. (22) and (23) reduce to power-law scaling in the limiting cases: \(D_{\text{u, bulk}} \gg D_{\text{u, BL}}\) and \(D_{\text{u, bulk}} \ll D_{\text{u, BL}}\). First, we will estimate the regimes of \(\text{Ra}\) and \(\text{Pr}\), where the viscous and thermal dissipation rates dominate in the bulk or in the boundary layers. Using \(f_i's\) and Eqs. (11)–(14), we deduce that

\[
\frac{D_{\text{u, BL}}}{D_{\text{u, bulk}}} = \frac{f_2}{f_1} \frac{d}{1} \frac{\text{Re}}{\text{Pr}}
\]

(46)

\[
\frac{D_{\text{T, BL}}}{D_{T, bulk}} = \frac{2f_4}{f_3} \frac{\text{Nu}}{\text{Re}\text{Pr}}
\]

(47)

In Figs. 9(a) and 9(b), we exhibit the plots of the above estimates for \(\text{Pr} = 0.02, 1,\) and 50. We also exhibit the numerically computed points in Fig. 9; these points are consistent with the estimates given by Eqs. (46) and (47). On the other hand, the ratio of the dissipation rates estimated using the GL model [by employing the bulk and the boundary layer terms of Eqs. (16) and (17)] deviates significantly from the numerically computed points.

The plots show that the thermal dissipation rate in the boundary layers exceeds that in the bulk by a factor of two to four for all \(\text{Pr}\) values. On the other hand, the viscous dissipation rate in the bulk exceeds that in the boundary layers for \(\text{Ra} \geq 10^5\). These observations are in agreement with previous studies. The plots imply that \(D_{\text{u, BL}}\) dominates \(D_{\text{u, bulk}}\) only for \(\text{Ra} \ll 10^5\), where \(\text{Nu} = 1\). However, recall that the power-law relations for this limiting case, given by Eqs. (30) and (31), are invalid for small \(\text{Nu}\). Thus, we do not examine this limiting case further.
For the regimes characterized by $D_{\text{bulk}} \gg D_{\text{BL}}$, we plug the best-fit relation for $f_i$ in Eqs. (27) and (28) to obtain the following:

\[
\begin{align*}
\text{Re} &= \begin{cases} 
0.76\text{Ra}^{0.42}\text{Pr}^{0.72}, & \text{Small Pr}, \\
0.20\text{Ra}^{0.50}\text{Pr}^{0.86}, & \text{Moderate Pr}, \\
0.11\text{Ra}^{0.55}\text{Pr}^{0.94}, & \text{Large Pr},
\end{cases} \\
\text{Nu} &= \begin{cases} 
0.30\text{Ra}^{0.22}\text{Pr}^{0.11}, & \text{Small Pr}, \\
0.21\text{Ra}^{0.29}\text{Pr}^{0.03}, & \text{Moderate Pr}, \\
0.21\text{Ra}^{0.30}\text{Pr}^{0.03}, & \text{Large Pr}.
\end{cases}
\end{align*}
\] (48)

Since $f_4$ is a very weak function of Ra and Pr, we assume it to be a constant ($\approx 0.37$). The Ra dependence described by Eqs. (48) and (49) is consistent with the scaling observed for large Rayleigh numbers ($10^8 \ll \text{Ra} \ll 10^{15}$) in the literature.\(^3,\text{11,16–18,22,23,36,37,61,71–73}\)

Furthermore, the above relation for Re and Nu in the small Pr regime is not very far from GL's predictions of $\text{Re} \sim \text{Ra}^{2/5}\text{Pr}^{3/5}$ and $\text{Nu} \sim \text{Ra}^{1/3}\text{Pr}^{1/3}$. The derived relation for Nu is also in agreement with analytically derived upper bounds of $\text{Nu} \leq \text{Ra}^{1/3}\ln(\text{Ra})^{7/4}$ and $\text{Nu} \leq 0.64\text{Ra}^{1/3}\text{ln}^{-1/3}$.\(^71\) Equation (49) also suggests that Nu is a weak function of Pr for moderate and large Pr [see Fig. 8(b)].

For very large Ra ($\gg 10^8$), some recent works\(^45,76\) reveal that the Nusselt number scales in the band from $\text{Nu} \lesssim \text{Ra}^{0.33}$ to $\text{Ra}^{0.35}$. Unfortunately, our predictions are not very accurate in this regime; this is because the functional forms of $f_i$ are constructed using data from simulations with $\text{Ra} \lesssim 10^{10}$. Note that for larger Ra, we expect the suppression of viscous and thermal dissipation rates to weaken because of the thin boundary layers. This can, in turn, cause the scaling exponent for Nu to increase. For example, $f_2$ and $f_3$ may scale as

\[f_2 \sim \text{Ra}^{-0.34}, \quad f_3 \sim \text{Ra}^{-0.16},\] (50)

instead of $\text{Ra}^{-0.21}$ as per Eqs. (41) and (43). Plugging the above expressions for $f_2$ and $f_3$ in Eq. (28) gives

\[\text{Nu} \sim \text{Ra}^{-0.33},\]

which is consistent with the results of Iyer et al.\(^45\) However, the scalings for $f_2$ and $f_3$, given by Eq. (50), are conjectures that need to be verified using simulations with large Ra’s. In a future work, we plan to upgrade our present work by taking inputs from large Ra simulations.

We conclude in Sec. V.

V. CONCLUSIONS

In this paper, we enhance Grossmann and Lohse’s model to provide improved predictions of Reynolds and Nusselt numbers in turbulent Rayleigh–Bénard convection. The process of obtaining this relation involves Grossman and Lohse’s idea of splitting the total viscous and thermal dissipation rates into bulk and boundary layer contributions and using the exact relations of Shraiman and Siggia. In the present work, we address the additional Ra and Pr dependences on the viscous and thermal dissipation rates in the bulk compared to free turbulence, as well as the deviation of viscous boundary layer thickness from Prandtl–Blasius theory.

The Reynolds and Nusselt numbers are obtained by solving a cubic polynomial equation consisting of four functions $f_i(\text{Ra, Pr})$ that are prefactors for the dissipation rates in the bulk and boundary layers. Note that these prefactors were constants in the original GL model. The aforementioned functions are determined using machine learning (regression analysis) on 60 datasets obtained from direct numerical simulations of RBC. The cubic polynomial equation reduces to power-law expressions in the limit of the viscous dissipation rate dominating in the bulk.

Using functional forms for the prefactors for the dissipation rates improves the predictions for both Re and Nu compared to the GL model. We observe significant improvements in the predictions of Re, which is important because Re is more sensitive to modeling parameters compared to Nu. The improvement in the predictions of Nu is more pronounced for extreme Pr regimes ($\text{Pr} \leq 0.02$ and $\geq 100$). Our results underscore the importance of applying data-driven methods to improve existing models, a practice that has recently been picking up pace in research on turbulence.

Presently, our work takes inputs from data that are restricted to $\text{Ra} < 10^{10}$ and unit aspect ratio. Our predictions can be further enhanced after determining $f_i$ for $\text{Ra} > 10^{10}$ and for different aspect ratios. Moreover, our work can be extended to convection with magnetic fields following the approach of Zürner et al.\(^74,76\)

We believe that our results will be valuable to the scientific and engineering community, especially where flows with extreme Prandtl numbers are involved. For example, they will help understand the fluid dynamics and heat transport in liquid metal batteries that involve small Pr convection.\(^31\) On the other end, our analysis will help strengthen our knowledge on mantle convection, which involves a large Pr flow.\(^1,2,82\) This will, in turn, enable us to make better predictions of seismic disturbances and the earth’s magnetic field. Apart from this, our present work should also aid in expanding our knowledge on oceanic and atmospheric flows and, thus, enable us to make improved weather predictions.

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APPENDIX: ROBUSTNESS OF THE ESTIMATED PARAMETER VALUES FOR $f_i(\text{Ra, Pr})$

In this section, we check the robustness of the parameter values for $f_i(\text{Ra, Pr})$ estimated in Sec. IV D. Toward this objective, we employ the regression algorithm, used in Sec. IV D, on a reduced training set consisting of 30 data points, which is half of the total number of data points, and test the algorithm on the remaining 30 data points. Starting from the point corresponding to Pr = 0.02 and Ra = $5 \times 10^8$, we take alternate data points from Tables I and II for training and the remaining data points for testing. We obtain the
following fits for $f_i$ for the reduced training set:

$$f_1 = 0.72 H_1 Pr_1^{0.30} + 28 H_2 Ra_1^{-0.21} Pr_1^{0.52} + 150 H_3 Ra_1^{-0.33} Pr_1^{0.79}, \quad (A1)$$

$$\frac{f_2}{\delta u} = 4.1 H_1 Ra_2^{0.26} Pr_2^{-0.27} + 6.9 H_2 Ra_2^{0.23} Pr_2^{-0.30} + 21 H_3 Ra_2^{0.15} Pr_2^{-0.18}, \quad (A2)$$

$$f_3 = 0.087 H_1 Ra_3^{-0.14} Pr_3^{-0.16} + 0.26 H_2 Ra_3^{-0.21} Pr_3^{-0.17} + 0.40 H_3 Ra_3^{-0.24} Pr_3^{-0.095}, \quad (A3)$$

$$f_4 = 0.45 H_1 Ra_4^{-0.012} Pr_4^{-0.0075} + 0.42 H_2 Ra_4^{-0.0078} Pr_4^{-0.0050} + 0.36 H_3 Pr_4^{0.0161}. \quad (A4)$$

We observe that the fits given by Eqs. (A1)–(A4) are similar to those of Eqs. (41)–(44), which correspond to the fits obtained when all the data points were used as training sets. The average deviation between the $f_i$‘s predicted by the fits and the actual values of the test set are 25%, 20%, 13%, and 71% for $f_1$, $f_2/\delta u$, $f_3$, and $f_4$, respectively. These deviations are almost the same as those observed when all the datasets were used for training and testing. Furthermore, if we train our algorithm using only 15 datasets (every fourth set from Tables I and II), we obtain

$$f_1 = 0.68 H_1 Pr_1^{0.31} + 25 H_2 Ra_1^{-0.20} Pr_1^{0.47} + 238 H_3 Ra_1^{-0.37} Pr_1^{0.81}, \quad (A5)$$

$$\frac{f_2}{\delta u} = 3.7 H_1 Ra_2^{0.26} Pr_2^{-0.27} + 5.8 H_2 Ra_2^{0.24} Pr_2^{-0.33} + 23 H_3 Ra_2^{0.15} Pr_2^{-0.19}, \quad (A6)$$

$$f_3 = 0.066 H_1 Ra_3^{-0.15} Pr_3^{-0.17} + 0.23 H_2 Ra_3^{-0.20} Pr_3^{-0.19} + 0.40 H_3 Ra_3^{-0.24} Pr_3^{-0.090}, \quad (A7)$$

$$f_4 = 0.42 H_1 Ra_4^{-0.0099} + 0.41 H_2 Ra_4^{-0.0069} Pr_4^{-0.0059} + 0.38 H_3, \quad (A8)$$

with the average deviation between the $f_i$‘s predicted by the fits and the actual values of the test set being 26%, 21%, 16%, and 71% for $f_1$, $f_2/\delta u$, $f_3$, and $f_4$, respectively. We observe that there are visible changes in the parameter values estimated using 15 datasets. Thus, we infer that the parameter values estimated using more than 30 datasets are reasonably robust.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

REFERENCES


