Introduction to Statstical Theory of Fluid Turbulence

Mahendra K. Verma*
Department of Physics, IIT Kanpur, Kanpur 208016, India

This is a brief introduction to the statistical theory of fluid turbulence, with emphasis on field-theoretic treatment of renormalized viscosity and energy fluxes.

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*Electronic address: mkv@iitk.ac.in; URL: http://home.iitk.ac.in/~mkv

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I. INTRODUCTION

Fluid and plasma flows exhibit complex random behaviour at high Reynolds number; this phenomena is called turbulence. On the Earth turbulence is observed in atmosphere, channel and rivers flows etc. In the universe, most of the astrophysical systems are turbulent. Some of the examples are solar wind, convective zone in stars, galactic plasma, accretion disk etc.

Reynolds number, defined as UL/ν (U is the large-scale velocity, L is the large length scale, and ν is the kinematic viscosity), has to be large (typically 2000 or more) for turbulence to set in. At large Reynolds number, there are many active modes which are nonlinearly coupled. These modes show random behaviour along with rich structures and long-range correlations. Presence of large number of modes and long-range correlations makes turbulence a very difficult problem that remains largely unsolved for more than hundred years.

Fortunately, random motion and presence of large number of modes make turbulence amenable to statistical analysis. Notice that the energy supplied at large-scales (L) gets dissipated at small scales, say l_d. Experiments and numerical simulations show that the velocity difference $u(\mathbf{x}+\mathbf{l}) - u(\mathbf{x})$ has a universal probability density function (pdf) for $l_d \ll l \ll L$. That is, the pdf is independent of experimental conditions, forcing and dissipative mechanisms etc. Because of its universal behaviour, the above quantity has been of major interest among physicists for last sixty years. Unfortunately, we do not yet know how to derive the form of this pdf from the first principle, but some of the moments have been computed analytically. The range of scales l satisfying $l_d \ll l \ll L$ is called inertial range.

In 1941 Kolmogorov [19, 20, 21] computed an exact expression for the third moment of velocity difference. He showed that under vanishing viscosity, third moment for velocity difference for homogeneous, isotropic, incompressible, and steady-state fluid turbulence is

$$\left\langle \left(u_{||}(\mathbf{x}+\mathbf{l}) - u_{||}(\mathbf{x})\right)^3 \right\rangle = \frac{4}{5}\Pi l$$

where $||$ is the parallel component along $\mathbf{l}$, $\langle . \rangle$ stands for ensemble average, and Π is the energy cascade rate, which is also equal to the energy supply rate at large scale L and dissipation rate at the small scale l_d. Assuming fractal structure for the velocity field, and Π to be constant for all l, we can show that the energy spectrum $E(k)$ is

$$E(k) = K_{Ko}\Pi^{2/3}k^{-5/3},$$

where K_{Ko} is a universal constant, called Kolmogorov's constant, and $L^{-1} \ll k \ll l_d^{-1}$. Numerical simulations and experiments verify the above energy spectrum apart from a small deviation called intermittency correction.

Availability of powerful computers and sophisticated theoretical tools have helped us understand several aspects of fluid turbulence. Some of these theories have been motivated by Kolmogorov's theory for fluid turbulence. Note that incompressible turbulence is better understood than compressible turbulence. Therefore, our discussion is primarily for incompressible plasma. *In this paper we focus on the universal statistical properties of fluid turbulence, which are valid in the inertial range. In this paper we will review the statistical properties of following quantities:*

1. Inertial-range energy spectrum for fluid turbulence.

2. Various energy fluxes in fluid turbulence.

3. Energy transfers between various wavenumber shells.

Many analytic calculations in fluid have been done using field-theoretic techniques. Even though these methods are plagued with some inconsistencies, many meaningful results have been obtained using them. Here we will discuss items 1-3 in greater detail.

As mentioned above, pdf of velocity difference in fluid turbulence is still unsolved. We know from experiments and simulation that pdf is close to Gaussian for small δu, but is nongaussian for large δu. This phenomena is called intermittency. Note that various moments called Structure functions are connected to pdf. It can be shown that the structure functions are related to the "local energy cascade rate" $\Pi(k)$. Some phenomenological models, notably by She and Leveque [43] based on log-Poisson process, have been developed to compute $\Pi(k)$; these models quite successfully capture intermittency in both fluid and MHD turbulence. The predictions of these models are in good agreement with numerical results. There are interesting recent advances in intermittency in scalar turbulence [**?**]. However, here we do not discuss intermittency in great detail.

Numerical simulations have provided many important data and clues for understanding the dynamics of turbulence. They have motivated new models, and have verified/rejected existing models. In that sense, they have become another type of experiment, hence commonly termed as numerical experiments. Modern computers have made reasonably high resolution simulations possible. The highest resolution simulation in fluid turbulence is on 4096^3 grid (e.g., by Gotoh [18]). Note that simulations are also used heavily for studying fluid flows around aircrafts and vehicles, in atmospheres, engineering devices like turbines etc.

Fluid turbulence has a larger volume of literature. Here we will list only some of the relevant ones. Leslie [30], McComb [33, 34, 35], Zhou et al. [63], and Smith and Woodruff [44] have reviewed field-theoretic treatment of fluid turbulence. The recent books by Frisch [17] and Lesieur [29] cover recent developments and phenomenological theories. Also, the review articles by Orszag [39], Kraichnan and Montgomery [27], and Sreenivasan [45] are quite exhaustive.

The outline of the paper is as follows: Section II contains definition of various global and spectral quantities along with their governing equations. In Section III we discuss the formalism of "mode-to-mode" energy transfer rates in fluid turbulence. Using this formalism, formulas for energy fluxes and shell-to-shell energy transfer rates have been derived. Section IV contains the existing discussion on Kolmogorov's phenomenology for fluid turbulence. Sections V and VI contains very brief introduction on experiments and simulations in fluid turbulence.

phenomenologies which include Kraichnan's 3/2 mode; Kolmogorov-like models of Goldreich and Sridhar. Absolute equilibrium theories and Selective decay are also discussed here. In Section V we review the observed energy spectra of the solar wind. Section VI describes Pseudo-spectral method along with the numerical results on energy spectra, fluxes, and shell-to-shell energy transfers. In these discussions we verify which of the turbulence phenomenologies are in agreement with the solar wind data and numerical results. In Section VII we introduce Renormalization-group analysis of turbulence, with an emphasis on McComb's procedure. In Section VIII we compute various energy fluxes and shell-to-shell energy transfers in fluid turbulence using field-theoretic techniques. Here we also review eddy-damped quasi-normal Markovian (EDQNM) calculations of turbulence.

Section IX contains a brief discussion on intermittency models of fluid turbulence. The last section (X) contains concluding remarks. Appendix A contains the definitions of Fourier series and transforms of fields in homogeneous turbulence. Appendix B contains the Feynman diagrams for fluid turbulence; these diagrams are used in the field-theoretic calculations. In the last two Appendix (C and D) we briefly mention the mode-to-mode energy transfer formalism for scalar and Rayleigh-Bénard convection.

II. GOVERNING EQUATIONS

A. Equations for Fluid Dynamics

Navier-Stokes Equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p_{th} + \nu\nabla^2\mathbf{u} + \frac{2\nu}{3}\nabla\nabla\cdot\mathbf{u}, \tag{1}$$

where p_{th} is called thermodynamic pressure. The law of mass conservation yields the following equation for density field $\rho(\mathbf{x})$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{2}$$

Pressure can be computed from ρ using the equation of state

$$p = f(\rho) \tag{3}$$

This completes the basic equations of fluids. Using these equations we can determine the unknowns ($\mathbf{u}$,ρ,p). Note that the number of equations and unknowns are the same.

On nondimensionalization of the Navier-Stokes equation, the term ∇p becomes $(d\rho/dx')/\rho \times (C_s/U)^2$, where C_s is the sound speed, U is the typical velocity of the flow, x' is the position coordinate normalized with relative to the length scale of the system [48]. $C_s \to \infty$ is the incompressible limit, which is widely studied because water, the most commonly found fluid on earth, is almost incompressible ($\delta\rho/\rho$ <0.01) in most practical situations. The other limit $C_s \to 0$ or $U \gg C_s$ (supersonic) is the fully compressible limit, and it is described by Burgers equation. The energy spectrum for both these extreme limits well known. When $U/C_s \ll 1$ but nonzero, then we call the fluid to be nearly incompressible; Zank and Matthaeus [59, 60] have given theories for this limit. The energy and density spectra are not well understood for arbitrary U/C_s.

For most part of this paper, we assume the fluid to be incompressible. In most of the terrestrial experiments, the speed of water or air is less than the sound speed. Hence, incompressibility is a good assumption that simplifies the calculations significantly. The incompressibility approximation can also be interpreted as the limit when volume of a fluid parcel will not change along its path, that is, $d\rho/dt = 0$. From the continuity equation (2), the incompressibility condition reduces to

$$\nabla\cdot\mathbf{u} = 0 \tag{4}$$

This is a constraint on the velocity field $\mathbf{u}$. Note that incompressibility does not imply constant density. However, for simplicity we take density to be constant and equal to 1. Under this condition, Eqs. (1, 2) reduce to

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u}, \tag{5}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{6}$$

When we take divergence of the equation Eq. (5), we obtain Poisson's equation

$$-\nabla^2 p = \nabla\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}]. \tag{7}$$

Hence, given $\mathbf{u}$ fields at any given time, we can evaluate p. Hence p is a dependent variable in the incompressible limit.

The Navier-Stokes equation is nonlinear, and that is the crux of the problem. The viscous ($\nu\nabla^2\mathbf{u}$) term dissipates the input energy. The ratio of the nonlinear vs. viscous dissipative term is called Reynolds number $Re = UL/\nu$, where U is the velocity scale, and L is the length scale. For turbulent flows, Reynolds number should be high, typically more than 2000 or so.

Table I: Global Quantities in MHD

Quantity	Symbol	Definition	Conserved in MHD?
Kinetic Energy	E	$\int d\mathbf{x}u^2/2$	No
Kinetic Helicity	H_K	$\int d\mathbf{x}(\mathbf{u}\cdot\omega)/2$	No
Enstrophy	Ω	$\int d\mathbf{x}\omega^2/2$	No

B. Energy Equations and Conserved Quantities

In this subsection we derive energy equations for compressible and incompressible fluids. For compressible fluids we can construct equations for energy using Eq. (1). Following Landau [28] we derive the following energy equation for the kinetic energy

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2+\rho\epsilon\right)=-\nabla\cdot\left[\left(\frac{1}{2}u^2+\epsilon\right)\rho\mathbf{u}\right]-\nabla\cdot p\mathbf{u}+\Phi \tag{8}$$

where ϵ is the internal energy function. The first term in the RHS is the energy flux, and the second term is the work done by the pressure, which enhances the energy of the system. The third term,Φ, a complex function of strain tensor, is the energy change due to surface forces.

In the above equations we apply isoentropic and incompressibility conditions. For the incompressible fluids we can choose $\rho=1$. Landau [28] showed that under this condition ϵ is a constant. Hence, for incompressible fluid we treat $(u^2)/2$ as total energy. For ideal incompressible fluid ($\nu=0$) the energy evolution equation is

$$\frac{\partial}{\partial t}\frac{1}{2}\left(u^2\right)=-\nabla\cdot\left[\left(\frac{1}{2}u^2+p\right)\mathbf{u}\right] \tag{9}$$

By applying Gauss law we find that

$$\frac{\partial}{\partial t}\int\frac{1}{2}\left(u^2\right)d\mathbf{x}=-\oint\left[\left(\frac{1}{2}u^2+p\right)\mathbf{u}\right]\cdot d\mathbf{S} \tag{10}$$

For the boundary condition $u_n=0$ or periodic boundary condition, the total energy $\int 1/2(u^2)$ is conserved.

There are some more important quantities in fluid turbulence. They are listed in Table I. Note that ω is the vorticity field. By following the same procedure described above, we can show that in addition to energy, H_K is conserved in 3D fluids, while Ω is conserved in 2D fluids [29, 30]. The conserved quantities play very important role in turbulence.

Turbulent flow contains many interacting "modes", and the solution cannot be written in a simple way. A popular approach to analyze the turbulent flows is to use statistical tools. We will describe below the application of statistical methods to turbulence.

C. Necessity for Statistical Theory of Turbulence

In turbulent fluid the field variables are typically random both in space and time. Hence the exact solutions given initial and boundary conditions will not be very useful even when they were available (they are not!). However statistical averages and probability distribution functions are reproducible in experiments under steady state, and they shed important light on the dynamics of turbulence. For this reason many researchers study turbulence statistically. The idea is to use the tools of statistical physics for understanding turbulence. Unfortunately, only systems at equilibrium or near equilibrium have been understood reasonably well, and a good understanding of nonequilibrium systems (turbulence being one of them) is still lacking.

The statistical description of turbulent flow starts by dividing the field variables into mean and fluctuating parts. Then we compute averages of various functions of fluctuating fields. There are three types are averages: ensemble, temporal, and spatial averages. Ensemble averages are computed by considering a large number of identical systems and taking averages at corresponding instants over all these systems. Clearly, ensemble averaging demands heavily in experiments and numerical simulations. So, we resort to temporal and/or spatial averaging. Temporal averages are computed by measuring the quantity of interest at a point over a long period and then averaging. Temporal averages make sense for steady flows. Spatial averages are computed by measuring the quantity of interest at various spatial points at a given time, and then averaging. Clearly, spatial averages are meaningful for homogeneous systems.

Steady-state turbulent systems are generally assumed to be ergodic, for which the temporal average is equal to the ensemble average [17].

Navier-Stokes equation, which is really Newton's equation, is invariant under Galilean transformation

$$x = x' + V_0 t' \tag{11}$$
$$t = t' \tag{12}$$

where V_0 is the velocity of the primed reference frame with relative to the laboratory frame. Clearly, we can eliminate mean velocity of the flow by going to the frame whose velocity is the same as mean velocity of the fluid. Throughout this paper we will work in this reference frame.

As discussed above, certain symmetries like homogeneity help us in statistical description. Formally, homogeneity indicates that the average properties do not vary with absolute position in a particular direction, but depends only on the separation between points. For example, a homogeneous two-point correlation function is

$$\langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle = C_{ij}(\mathbf{x}-\mathbf{x}',t) = C_{ij}(\mathbf{r},t). \tag{13}$$

Similarly, *s*tationarity or steady-state implies that average properties depend on time difference, not on the absolute time. That is,

$$\langle u_i(\mathbf{x},t)u_j(\mathbf{x},t')\rangle = C_{ij}(\mathbf{x},t-t'). \tag{14}$$

Another important symmetry is isotropy. A system is said to be isotropic if its average properties are invariant under rotation. For isotropic systems

$$\langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle = C_{ij}(|\mathbf{x}-\mathbf{x}'|,t) = C_{ij}(|\mathbf{r}|,t). \tag{15}$$

Isotropy reduces the number of independent correlation functions. Batchelor [1] showed that the isotropic two-point correlation could be written as

$$C_{ij}(\mathbf{r}) = C^{(1)}(r)r_i r_j + C^{(2)}(r)\delta_{ij} \tag{16}$$

where $C^{(1)}$ and $C^{(2)}$ are even functions of $r = |\mathbf{r}|$. Hence we have reduced the independent correlation functions to two. For incompressible flows, there is only one independent correlation function [1].

In turbulent fluid, fluctuations of all scales exist. Therefore, it is quite convenient to use Fourier basis for the representation of turbulent fluid velocity and magnetic field. Note that in recent times another promising basis called wavelet is becoming popular. In this paper we focus our attention on Fourier expansion, which is the topic of the next subsection.

D. Turbulence Equations in Spectral Space

Turbulent fluid velocity $\mathbf{u}(\mathbf{x},t)$ is represented in Fourier space as

$$\mathbf{u}(\mathbf{x},t) = \int \frac{d\mathbf{k}}{(2\pi)^d}\mathbf{u}(\mathbf{k},t)\exp(i\mathbf{k}\cdot\mathbf{x}) \tag{17}$$
$$\mathbf{u}(\mathbf{k},t) = \int d\mathbf{x}\mathbf{u}(\mathbf{x},t)\exp(-i\mathbf{k}\cdot\mathbf{x}) \tag{18}$$

where d is the space dimensionality.

In Fourier space, the equation for *incompressible* fluid is [3]

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)u_i(\mathbf{k},t) = -ik_i p(\mathbf{k},t) - ik_j\int\frac{d\mathbf{p}}{(2\pi)^d}u_j(\mathbf{k}-\mathbf{p},t)u_i(\mathbf{p},t) \tag{19}$$

with the following constrains

$$\mathbf{k}\cdot\mathbf{u}(\mathbf{k}) = 0, \tag{20}$$

The substitution of the incompressibility condition $\mathbf{k}\cdot\mathbf{u}(\mathbf{k}) = 0$ into Eq. (19) yields the following expression for the pressure field

$$p(\mathbf{k}) = -\frac{k_i k_j}{k^2}\int\frac{d\mathbf{p}}{(2\pi)^d}\left[u_j(\mathbf{k}-\mathbf{p},t)u_i(\mathbf{p},t)\right]. \tag{21}$$

Table II: Various Spectra of Fluid Turbulence

Quantity	Symbol	Derived from	Symbol for 1D
Kinetic energy spectrum	$C(\mathbf{k})$	$\langle u_i(\mathbf{k})u_j(\mathbf{k}')\rangle$	$E(k)$
Enstrophy spectrum	$\Omega(\mathbf{k})$	$\langle \omega_i(\mathbf{k})\omega_j(\mathbf{k}')\rangle$	$\Omega(k)$

Note that the density field has been taken to be a constant, and has been set equal to 1.

It is also customary to write the evolution equations symmetrically in terms of $\mathbf{p}$ and $\mathbf{k}-\mathbf{p}$ variables. The symmetrized equations are

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)u_i(\mathbf{k},t) = -\frac{i}{2}P^+_{ijm}(\mathbf{k})\int\frac{d\mathbf{p}}{(2\pi)^d}[u_j(\mathbf{p},t)u_m(\mathbf{k}-\mathbf{p},t)] \tag{22}$$

where

$$\begin{aligned} P^+_{ijm}(\mathbf{k}) &= k_jP_{im}(\mathbf{k})+k_mP_{ij}(\mathbf{k}); \\ P_{im}(\mathbf{k}) &= \delta_{im}-\frac{k_ik_m}{k^2}; \end{aligned}$$

Energy and other second-order quantities play important roles in fluid turbulence. For a homogeneous system these quantities are defined as

$$\langle u_i(\mathbf{k},t)u_j(\mathbf{k}',t)\rangle = C_{ij}(\mathbf{k},t)(2\pi)^d\delta(\mathbf{k}+\mathbf{k}').$$

The spectrum is also related to correlation function in real space

$$C_{ij}(\mathbf{r}) = \int\frac{d\mathbf{k}}{(2\pi)^d}C_{ij}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}).$$

For isotropic situations we can take $C_{ij}(\mathbf{k})$ to be an isotropic tensor, and it can be written as [1]

$$C_{ij}(\mathbf{k}) = P_{ij}(\mathbf{k})C(k). \tag{23}$$

When turbulence is isotropic, then a quantity called 1D spectrum or reduced spectrum $E(k)$ defined below is very useful.

$$\begin{aligned} E=\frac{1}{2}\langle u^2\rangle &= \tfrac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^d}C_{ii}(\mathbf{k}) \\ \int E(k)dk &= \frac{1}{2}\int dk\frac{S_dk^{d-1}}{(2\pi)^d}P_{ii}(\mathbf{k})C(\mathbf{k}) \\ &= \int dk\frac{S_dk^{d-1}(d-1)}{2(2\pi)^d}C(\mathbf{k}), \end{aligned}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of $d-$dimensional unit sphere. Therefore,

$$E(k) = C(\mathbf{k})k^{d-1}\frac{S_d(d-1)}{2(2\pi)^d}. \tag{24}$$

Note that the above formula is valid only for isotropic turbulence. We have tabulated all the important spectra of fluid turbulence in Table II.

The global quantities defined in Table I are related to the spectra defined in Table II by Perceval's theorem [1]. Since the fields are homogeneous, Fourier integrals are not well defined. In Appendix A we show that energy spectra defined using correlation functions are still meaningful because correlation functions vanish at large distances. We consider energy per unit volume, which are finite for homogeneous turbulence. As an example, the kinetic energy per unit volume is related to energy spectrum in the following manner:

$$\frac{1}{L^d}\int d\mathbf{x}\frac{1}{2}\langle u^2\rangle = \frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^d}C_{ii}(\mathbf{k}) = \int E(k)dk$$

Similar identities can be written for other fields.

In three dimensions we have another important quantities called kinetic helicities. In Fourier space kinetic helicity $H_K(\mathbf{k})$ is defined using

$$\langle u_i(\mathbf{k},t)\,\Omega_j(\mathbf{k}',t)\rangle = P_{ij}(\mathbf{k})\,H_K(\mathbf{k})(2\pi)^d\delta(\mathbf{k}+\mathbf{k}')$$

The total kinetic helicity H_K can be written in terms of

$$\begin{aligned} H_K &= \frac{1}{2}\langle \mathbf{u}(\mathbf{x})\cdot\mathbf{\Omega}(\mathbf{x})\rangle \\ &= \tfrac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^d}\frac{d\mathbf{k}'}{(2\pi)^d}\langle \mathbf{u}(\mathbf{k})\cdot\mathbf{\Omega}(\mathbf{k}')\rangle \\ &= \int \frac{d\mathbf{k}}{(2\pi)^d}H_K(\mathbf{k}) \\ &= \int dk H_K(k) \end{aligned}$$

Therefore, one dimensional magnetic helicity H_M is

$$H_K(k) = \frac{4\pi k^2}{(2\pi)^3}H_K(\mathbf{k}).$$

Using the definition $\mathbf{\Omega}(\mathbf{k}) = i\mathbf{k}\times\mathbf{u}(\mathbf{k})$, we obtain

$$\langle u_i(\mathbf{k},t)\,u_j(\mathbf{k}',t)\rangle = \left[P_{ij}(\mathbf{k})\,C^{uu}(\mathbf{k}) - i\epsilon_{ijl}k_l\frac{H_K(\mathbf{k})}{k^2}\right](2\pi)^d\delta(\mathbf{k}+\mathbf{k}').$$

Note that the magnetic helicity breaks mirror symmetry.

We can Fourier transform in time as well using

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) &= \int d\hat{k}\mathbf{u}(\mathbf{k},\omega)\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t) \\ \mathbf{u}(\mathbf{k},\omega) &= \int d\mathbf{x}dt\mathbf{u}(\mathbf{x},t)\exp(-i\mathbf{k}\cdot\mathbf{x}+i\omega t) \end{aligned}$$

where $d\hat{k} = d\mathbf{k}d\omega/(2\pi)^{d+1}$. The resulting fluid equations in $\hat{k} = (\mathbf{k},\omega)$ space are

$$\left(-i\omega+\nu k^2\right)u_i(\hat{k}) = -\frac{i}{2}P^+_{ijm}(\mathbf{k})\int_{\hat{p}+\hat{q}=\hat{k}} d\hat{p}\left[u_j(\hat{p})u_m(\hat{q})\right], \tag{25}$$

After we have introduced the energy spectra and other second-order correlation functions, we move on to discuss their evolution.

E. Energy Equations

The energy equation for general (compressible) Navier-Stokes is quite complex. However, incompressible Navier-Stokes equation is relatively simpler, and is discussed below.

From the evolution equations of fields, we can derive the following spectral evolution equations for incompressible MHD

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)C(\mathbf{k},t) = \frac{2}{(d-1)\,\delta(\mathbf{k}+\mathbf{k}')}\int_{\mathbf{k}'+\mathbf{p}+\mathbf{q}=\mathbf{0}}\frac{d\mathbf{p}}{(2\pi)^{2d}}[-\Im\langle(\mathbf{k}'\cdot\mathbf{u}(\mathbf{q}))\,(\mathbf{u}(\mathbf{p})\cdot\mathbf{u}(\mathbf{k}'))\rangle \tag{26}$$

where $\Im$ stands for the imaginary part. Note that $\mathbf{k}'+\mathbf{p}+\mathbf{q}=\mathbf{0}$ and $\mathbf{k}'=-\mathbf{k}$. In Eq. (26) the first term in the RHS provides the energy transfer from the velocity modes to $\mathbf{u}(\mathbf{k})$ mode. Note that pressure couples with compressible modes, hence it is absent in the incompressible equations.

In a finite box, using $\left\langle|\mathbf{u}(\mathbf{k})|^2\right\rangle = C(\mathbf{k})/((d-1)L^d)$, and $\delta(\mathbf{k})(2\pi)^d = L^d$ (see Appendix A), we can show that

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)\frac{1}{2}\left\langle|\mathbf{u}(\mathbf{k})|^2\right\rangle = \sum_{\mathbf{k}'+\mathbf{p}+\mathbf{q}=0}[-\Im\langle(\mathbf{k}'\cdot\mathbf{u}(\mathbf{q}))\,(\mathbf{u}(\mathbf{p})\cdot\mathbf{u}(\mathbf{k}'))\rangle],$$

Many important quantities, e.g. energy fluxes, can be derived from the energy equations. We will discuss these quantities in the next section.

III. MODE-TO-MODE ENERGY TRANSFERS AND FLUXES IN MHD TURBULENCE

In turbulence energy exchange takes place between various Fourier modes because of nonlinear interactions. Basic interactions in turbulence involves a wavenumber triad $(\mathbf{k}', \mathbf{p}, \mathbf{q})$ satisfying $\mathbf{k}' + \mathbf{p} + \mathbf{q} = \mathbf{0}$. Usually, energy gained by a mode in the triad is computed using the *combined energy transfer* from the other two modes [29]. Recently Dar et al. [7] devised a new scheme to compute the energy transfer rate between two modes in a triad, and called it *"mode-to-mode energy transfer"*. They computed energy cascade rates and energy transfer rates between two wavenumber shells using this scheme. We will review these ideas in this section. Note that we are considering only the interactions of incompressible modes.

A. "Mode-to-Mode" Energy Transfer in Fluid Turbulence

In this subsection we discuss evolution of energy in turbulent fluid *in a periodic box*. We consider ideal case where viscous dissipation is zero ($\nu = 0$). The equations are given in Lesieur [29]

$$\frac{\partial}{\partial t}\frac{1}{2}\left|u\left(\mathbf{k}'\right)\right|^2 = \sum_{\mathbf{k}'+\mathbf{p}+\mathbf{q}=0} -\frac{1}{2}\Im\left[\left(\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})\right)\left(\mathbf{u}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{p})\right) + \left(\mathbf{k}'\cdot\mathbf{u}(\mathbf{p})\right)\left(\mathbf{u}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{q})\right)\right], \tag{27}$$

where $\Im$ denotes the imaginary part. Note that the pressure does not appear in the energy equation because of the incompressibility condition.

Consider a case in which only three modes $\mathbf{u}(\mathbf{k}'), \mathbf{u}(\mathbf{p}), \mathbf{u}(\mathbf{q})$, and their conjugates are nonzero. Then the above equation yields

$$\frac{\partial}{\partial t}\frac{1}{2}\left|u\left(\mathbf{k}'\right)\right|^2 = \frac{1}{2}S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q}), \tag{28}$$

where

$$S(\mathbf{k}'|\mathbf{p},\mathbf{q}) = -\Im\left[\left(\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})\right)\left(\mathbf{u}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{p})\right) + \left(\mathbf{k}'\cdot\mathbf{u}(\mathbf{p})\right)\left(\mathbf{u}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{q})\right)\right]. \tag{29}$$

Lesieur and other researchers physically interpreted $S(\mathbf{k}'|\mathbf{p},\mathbf{q})$ as the *combined energy transfer rate* from modes $\mathbf{p}$ and $\mathbf{q}$ to mode $\mathbf{k}'$. The evolution equations for $|u(\mathbf{p})|^2$ and $|u(\mathbf{q})|^2$ are similar to that for $|u(\mathbf{k}')|^2$. By adding the energy equations for all three modes, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\left[\left|u\left(\mathbf{k}'\right)\right|^2 + \left|u\left(\mathbf{p}\right)\right|^2 + \left|u\left(\mathbf{q}\right)\right|^2\right]/2 &= S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q}) + S^{uu}(\mathbf{p}|\mathbf{q},\mathbf{k}') + S^{uu}(\mathbf{q}|\mathbf{k}',\mathbf{p}) && (30)\\
&= \Im[\left(\mathbf{q}\cdot\mathbf{u}(\mathbf{q})\right)\left(\mathbf{u}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{p})\right) && (31)\\
&\quad + \left(\mathbf{p}\cdot\mathbf{u}(\mathbf{p})\right)\left(\mathbf{u}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{q})\right) && (32)\\
&\quad + \left(\mathbf{k}'\cdot\mathbf{u}(\mathbf{k}')\right)\left(\mathbf{u}(\mathbf{p})\cdot\mathbf{u}(\mathbf{q})\right)] && (33)
\end{aligned}$$

For incompressible fluid the right-hand-side is identically zero because $\mathbf{k}'\cdot\mathbf{u}(\mathbf{k}') = 0$. Hence the energy in each interacting triad is conserved , i.e.,

$$\left|u\left(\mathbf{k}'\right)\right|^2 + \left|u\left(\mathbf{p}\right)\right|^2 + \left|u\left(\mathbf{q}\right)\right|^2 = const. \tag{34}$$

The question is whether we can derive an expression for mode-to-mode energy transfer rates from mode $\mathbf{p}$ to mode $\mathbf{k}'$, and from mode $\mathbf{q}$ to mode $\mathbf{k}'$ separately. Dar et al. [7] showed that it is meaningful to talk about energy transfer rate between two modes. They derived an expression for the mode-to-mode energy transfer, and showed it to be unique apart from an irrelevant arbitrary constant. They referred to this quantity as "mode-to-mode energy transfer". Even though they talk about mode-to-mode transfer, they are still within the framework of triad interaction, i.e., a triad is still the fundamental entity of interaction.

1. *Definition of Mode-to-Mode Transfer in a Triad*

Consider a triad $(\mathbf{k}', \mathbf{p}, \mathbf{q})$. Let the quantity $R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ denote the energy transferred from mode $\mathbf{p}$ to mode $\mathbf{k}'$ with mode $\mathbf{q}$ playing the role of a mediator. We wish to obtain an expression for R.

The R's should satisfy the following relationships :

1. The sum of energy transfer from mode $\mathbf{p}$ to mode $\mathbf{k}'$ ($R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$**),** and from mode $\mathbf{q}$ to mode $\mathbf{k}'$ ($R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$) should be equal to the total energy transferred to mode $\mathbf{k}'$ from modes $\mathbf{p}$ and $\mathbf{q}$, i.e., $S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q})$ [see Eq. (28)]. That is,

$$R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + R^{uu}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) = S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q}), \tag{35}$$

$$R^{uu}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) + R^{uu}(\mathbf{p}|\mathbf{q}|\mathbf{k}') = S^{uu}(\mathbf{p}|\mathbf{k}',\mathbf{q}), \tag{36}$$

$$R^{uu}(\mathbf{q}|\mathbf{k}'|\mathbf{p}) + R^{uu}(\mathbf{q}|\mathbf{p}|\mathbf{k}') = S^{uu}(\mathbf{q}|\mathbf{k}',\mathbf{p}). \tag{37}$$

2. By definition, the energy transferred from mode $\mathbf{p}$ to mode $\mathbf{k}'$, $R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$, will be equal and opposite to the energy transferred from mode $\mathbf{k}'$ to mode $\mathbf{p}$, $R^{uu}(\mathbf{p}|\mathbf{k}'|\mathbf{q})$. Thus,

$$R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + R^{uu}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) = 0, \tag{38}$$

$$R^{uu}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) + R^{uu}(\mathbf{q}|\mathbf{k}'|\mathbf{p}) = 0, \tag{39}$$

$$R^{uu}(\mathbf{p}|\mathbf{q}|\mathbf{k}') + R^{uu}(\mathbf{q}|\mathbf{p}|\mathbf{k}') = 0. \tag{40}$$

These are six equations with six unknowns. However, the value of the determinant formed from the Eqs. (35-40) is zero. Therefore we cannot find unique R's given just these equations. In the following discussion we will study the set of solutions of the above equations.

2. Solutions of equations of mode-to-mode transfer

Consider a function

$$S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = -\Im\left([\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})]\,[\mathbf{u}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{p})]\right) \tag{41}$$

Note that $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ is altogether different function compared to $S(\mathbf{k}'|\mathbf{p},\mathbf{q})$. In the expression for $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$, the field variables with the first and second arguments are dotted together, while the field variable with the third argument is dotted with the first argument.

It is very easy to check that $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ satisfy the Eqs. (35-40). Note that $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ satisfy the Eqs. (38-40) because of incompressibility condition. The above results implies that the set of $S^{uu}(||)$'s is *one instance* of the $R^{uu}(||)$'s. However, $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ is not a unique solution. If another solution $R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ differs from $S(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ by an arbitrary function X_Δ, i.e., $R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + X_\Delta$, then by inspection we can easily see that the solution of Eqs. (35-40) must be of the form

$$R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + X_\Delta \tag{42}$$

$$R^{uu}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) = S^{uu}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) - X_\Delta \tag{43}$$

$$R^{uu}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) = S^{uu}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) - X_\Delta \tag{44}$$

$$R^{uu}(\mathbf{p}|\mathbf{q}|\mathbf{k}') = S^{uu}(\mathbf{p}|\mathbf{q}|\mathbf{k}') + X_\Delta \tag{45}$$

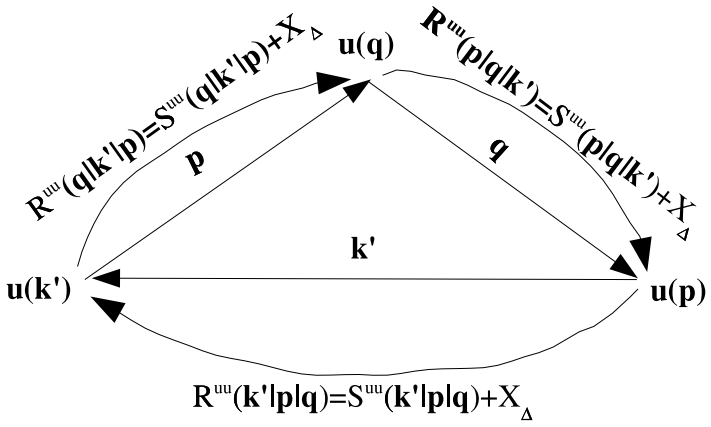


Figure 1: Mode-to-mode energy transfer in fluid turbulence. S^{uu}(**k'**|**p**|**q**) represents energy transfer rate from mode $\mathbf{u}(\mathbf{p})$ to mode $\mathbf{u}(\mathbf{k}')$ with the mediation of mode $\mathbf{u}(\mathbf{q})$. X_Δ is the arbitrary circulating transfer.

$$R^{uu}(\mathbf{q}|\mathbf{k}'|\mathbf{p}) = S^{uu}(\mathbf{q}|\mathbf{k}'|\mathbf{p}) + X_\Delta \tag{46}$$

$$R^{uu}(\mathbf{q}|\mathbf{p}|\mathbf{k}') = S^{uu}(\mathbf{q}|\mathbf{p}|\mathbf{k}') - X_\Delta \tag{47}$$

Hence, the solution differs from $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ by a constant. See Fig. 1 for illustration. A careful observation of the figure indicates that the quantity X_Δ flows along $\mathbf{p} \to \mathbf{k}' \to \mathbf{q} \to \mathbf{p}$, circulating around the entire triad without changing the energy of any of the modes. Therefore we will call it the *Circulating transfer*. Of the total energy transfer between two modes, $S^{uu} + X_\Delta$, only S^{uu} can bring about a change in modal energy. X_Δ transferred from mode $\mathbf{p}$ to mode $\mathbf{k}'$ is transferred back to mode $\mathbf{p}$ via mode $\mathbf{q}$. Thus the energy that is effectively transferred from mode $\mathbf{p}$ to mode $\mathbf{k}'$ is just $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$. Therefore $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ can be termed as the *effective mode-to-mode energy transfer* from mode $\mathbf{p}$ to mode $\mathbf{k}'$.

Note that X_Δ can be a function of wavenumbers $\mathbf{k}', \mathbf{p}, \mathbf{q}$, and the Fourier components $\mathbf{u}(\mathbf{k}'), \mathbf{u}(\mathbf{p}), \mathbf{u}(\mathbf{q})$. It may be possible to determine X_Δ using constraints based on invariance, symmetries, etc. Dar et al. [6] attempted to obtain X_Δ using this approach, but could show that X_Δ is zero to linear order in the expansion. However, a general solution for X_Δ could not be found. Unfortunately, X_Δ cannot be calculated even by simulation or experiment, because we can experimentally compute only the energy transfer rate to a mode, which is a sum of two mode-to-mode energy transfers. The mode-to-mode energy transfer rate is really an abstract quantity, somewhat similar to "gauges" in electrodynamics.

The terms $u_j \partial_j u_i$ and $u_i u_j \partial_j u_i$ are nonlinear terms in the Navier-Stokes equation and the energy equation respectively. When we look at the formula (41) carefully, we find that the $u_j(\mathbf{q})$ term is contracted with k_j in the formula. Hence, u_j field is the mediator in the energy exchange between first (u_i) and third field (u_i) of $u_i u_j \partial_j u_i$.

In this following discussion we will compute the energy fluxes and the shell-to-shell energy transfer rates using $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$.

B. Shell-to-Shell Energy Transfer in Fluid Turbulence Using Mode-to-mode Formalism

In turbulence energy transfer takes place from one region of the wavenumber space to another region. Domaradzki and Rogallo [10] have discussed the energy transfer between two shells using the combined energy transfer S^{uu}(**k'**|**p,q**).

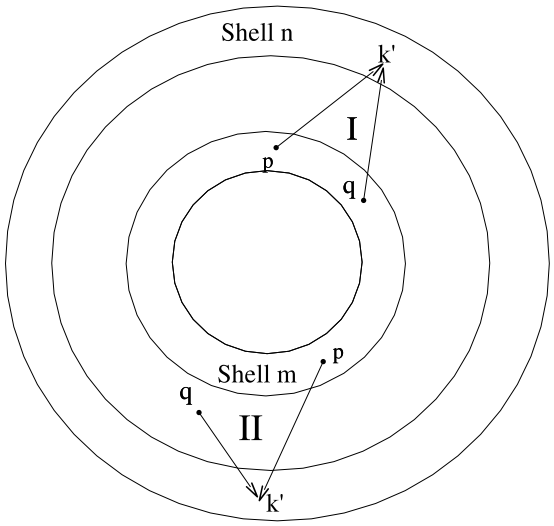


Figure 2: Shell-to-shell energy transfer from wavenumber-shell m to wavenumber-shell n. The triads involved in this process fall in two categories: Type I, where both $\mathbf{p}$ and $\mathbf{q}$ are inside shell m, and Type II, where only $\mathbf{p}$ is inside shell m.

They interpret the quantity

$$T_{nm}^{uu} = \frac{1}{2} \sum_{\mathbf{k}' \in n} \sum_{\mathbf{p} \in m} S^{uu}(\mathbf{k}'|\mathbf{p}, \mathbf{q}). \tag{48}$$

as the rate of energy transfer from shell m to shell n . Note that $\mathbf{k}'$-sum is over shell n, $\mathbf{p}$-sum over shell m, and $\mathbf{k}' + \mathbf{p} + \mathbf{q} = 0$. However, Domaradzki and Rogallo [10] themselves points out that it may not be entirely correct to interpret the formula (48) as the shell-to-shell energy transfer. The reason for this is as follows.

In the energy transfer between two shells m and n, two types of wavenumber triads are involved, as shown in Fig. 2. The real energy transfer from the shell m to the shell n takes place through both $\mathbf{k}'$-$\mathbf{p}$ and $\mathbf{k}'$-$\mathbf{q}$ legs of triad I, but only through $\mathbf{k}'$-$\mathbf{p}$ leg of triad II. But in Eq. (48) summation erroneously includes $\mathbf{k}'$-$\mathbf{q}$ leg of triad II also along with the three legs given above. Hence Domaradzki and Ragallo's formalism [10] do not yield totally correct shell-to-shell energy transfers, as was pointed out by Domaradzki and Rogallo themselves. We will show below how Dar et al.'s formalism [7] overcomes this difficulty.

By definition of the the mode-to-mode transfer function $R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$, the energy transfer from shell m to shell n can be defined as

$$T_{nm}^{uu} = \sum_{\mathbf{k}' \in n} \sum_{\mathbf{p} \in m} R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) \tag{49}$$

where the $\mathbf{k}'$-sum is over the shell n, and $\mathbf{p}$-sum is over the shell m. The quantity R^{uu} can be written as a sum of an effective transfer $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ and a circulating transfer X_Δ. As discussed in the last section, the circulating transfer X_Δ does not contribute to the energy change of modes. From Figs. 1 and 2 we can see that X_Δ flows from the shell m to the shell n and then flows back to m indirectly through the mode $\mathbf{q}$. Therefore the *effective* energy transfer from the shell m to the shell n is just $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ summed over all the $\mathbf{k}'$-modes in the shell n and all the $\mathbf{p}$-modes in the shell m, i.e.,

$$T_{nm}^{uu} = \sum_{\mathbf{k}' \in n} \sum_{\mathbf{p} \in m} S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}). \tag{50}$$

Clearly, the energy transfer through $\mathbf{k}' - \mathbf{q}$ of the triad II of Fig. 2 is not present in T_{nm}^{uu} in Dar et al.'s formalism because $\mathbf{q} \notin m$. Hence, the formalism of the mode-to-mode energy transfer rates provides us a correct and convenient method to compute the shell-to-shell energy transfer rates in fluid turbulence.

C. Energy Cascade Rates in Fluid Turbulence Using Mode-to-mode Formalism

The kinetic energy cascade rate (or flux) Π in fluid turbulence is defined as the rate of loss of kinetic energy by the modes inside a sphere to the modes outside the sphere. Let k_0 be the radius of the sphere under consideration. Kraichnan [23], Leslie [30], and others have computed the energy flux in fluid turbulence using $S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q})$

$$\Pi(k_0) = -\sum_{|\mathbf{k}|<k_0}\sum_{|\mathbf{p}|>k_0}\frac{1}{2}S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q}). \tag{51}$$

Although the energy cascade rate in fluid turbulence can be found by the above formula, the mode-to-mode approach of Dar et al. [7] provides a more natural way of looking at the energy flux. Since $R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ represents energy transfer from $\mathbf{p}$ to $\mathbf{k}'$ with $\mathbf{q}$ as a mediator, we may alternatively write the energy flux as

$$\Pi(k_0) = \sum_{|\mathbf{k}'|>k_0}\sum_{|\mathbf{p}|<k_0} R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}). \tag{52}$$

However, $R^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + X_\Delta$, and the circulating transfer X_Δ makes no contribution to the energy flux from the sphere because the energy lost from the sphere through X_Δ returns to the sphere. Hence,

$$\Pi(k_0) = \sum_{|\mathbf{k}'|>k_0}\sum_{|\mathbf{p}|<k_0} S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}). \tag{53}$$

Both the formulas given above, Eqs. (51) and (53), are equivalent as shown by Dar et al. [6].

Frisch [17] has derived a formula for energy flux as

$$\Pi(k_0) = \left\langle \mathbf{u}_{\mathbf{k_0}}^{<} \cdot \left(\mathbf{u}_{\mathbf{k_0}}^{<} \cdot \nabla \mathbf{u}_{\mathbf{k_0}}^{>}\right)\right\rangle + \left\langle \mathbf{u}_{\mathbf{k_0}}^{<} \cdot \left(\mathbf{u}_{\mathbf{k_0}}^{>} \cdot \nabla \mathbf{u}_{\mathbf{k_0}}^{>}\right)\right\rangle. \tag{54}$$

It is easy to see that the above formula is consistent with mode-to-mode formalism. As discussed in the Subsection III A 2, the second field of both the terms are mediators in the energy transfer. Hence in mode-to-mode formalism, the above formula will translate to

$$\Pi(k_0) = \sum_{k>k_0}\sum_{p<k_0} -\Im\left[\left(\mathbf{k}'\cdot\mathbf{u}^{<}(\mathbf{q})\right)\left(\mathbf{u}^{<}(\mathbf{p})\cdot\mathbf{u}^{>}(\mathbf{k}')\right) + \left(\mathbf{k}'\cdot\mathbf{u}^{>}(\mathbf{q})\right)\left(\mathbf{u}^{<}(\mathbf{p})\cdot\mathbf{u}^{>}(\mathbf{k}')\right)\right],$$

which is same as mode-to-mode formula (53) of Dar et al. [7].

The above quantities are computed numerically or theoretically.

D. Digression to Infinite Box

In the above discussion we assumed that the fluid is contained in a finite volume. In simulations, box size is typically taken to 2π. However, most analytic calculations assume infinite box. It is quite easy to transform the equations given above to those for infinite box using the method described in Appendix. Here, the evolution of energy spectrum is given by (see Section II)

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) C\left(\mathbf{k},t\right) \;=\; \frac{2}{(d-1)\,\delta\left(\mathbf{k}+\mathbf{k}'\right)}\int_{\mathbf{k}'+\mathbf{p}+\mathbf{q}=\mathbf{0}} \frac{d\mathbf{p}}{(2\pi)^{2d}}\left[S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})\right] \tag{55}$$

The shell-to-shell energy transfer rate T_{nm} from the m-th shell to the n-th shell is

$$T_{nm} = \frac{1}{(2\pi)^d\delta(\mathbf{k}'+\mathbf{p}+\mathbf{q})}\int_{k'\in n}\frac{d\mathbf{k}'}{(2\pi)^d}\int_{p\in m}\frac{d\mathbf{p}}{(2\pi)^d}\left\langle S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})\right\rangle, \tag{56}$$

In terms of Fourier transform, the energy cascade rate from a sphere of radius $k_0$is

$$\Pi(k_0) = \frac{1}{(2\pi)^d\delta(\mathbf{k}'+\mathbf{p}+\mathbf{q})}\int_{k>k_0}\frac{d\mathbf{k}'}{(2\pi)^d}\int_{p<p_0}\frac{d\mathbf{p}}{(2\pi)^d}\left\langle S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})\right\rangle. \tag{57}$$

For isotropic flows, after some manipulation and using Eq. (24), we obtain [29]

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k,t) = T(k,t), \tag{58}$$

where $T(k,t)$, called *transfer function*, can be written in terms of $S^{YX}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$. The above formulas will be used in analytic calculations.

The mode-to-mode formalism discussed here is quite general, and it can be applied to scalar turbulence [52], MHD turbulence, Rayleigh-Bénard convection, enstrophy, Electron MHD etc. Some of these issues are discussed in Appendices C and D. One key assumption however is incompressibility. In the next section we will discuss various turbulence phenomenologies and models of fluid turbulence.

IV. TURBULENCE PHENOMENOLOGICAL MODELS

In the last two sections we introduced Navier-Stokes equation, and spectral quantities like the energy spectra and fluxes. These quantities have been analyzed using (a) phenomenological (b) numerical (c) analytical (d) experimental methods. In the present section we will present the most important phenomenological model called Kolmogorov's phenomenology of turbulence.

A. Kolmogorov's 1941 Theory for Fluid Turbulence

For homogeneous, isotropic, incompressible, and steady fluid turbulence with vanishing viscosity (large Re), Kolmogorov [19, 20, 21, 28] derived an exact relation that

$$\left\langle (\triangle u)_{\parallel}^3 \right\rangle = -\frac{4}{5}\epsilon l \tag{59}$$

where $(\triangle u)_{||}$ is component of $\mathbf{u}(\mathbf{x}+\mathbf{l}) - \mathbf{u}(\mathbf{x})$ along $\mathbf{l}$, ϵ is the dissipation rate, and l lies between forcing scale (L) and dissipative scales (l_d), i.e., $l_d \ll l \ll L$. This intermediate range of scales is called inertial range. Note that the above relationship is universal, which holds independent of forcing and dissipative mechanisms, properties of fluid (viscosity), and initial conditions. Therefore it finds applications in wide spectrum of phenomena, e. g., atmosphere, ocean, channels, pipes, and astrophysical objects like stars, accretion disks etc.

More popular than Eq. (59) is its equivalent statement on energy spectrum. If we assume $\triangle u$ to be fractal, and ϵ to be independent of scale, then

$$\left\langle (\triangle u)^2 \right\rangle \propto \epsilon^{2/3} l^{2/3} \tag{60}$$

Fourier transform of the above equation yields

$$E(k) = K_{Ko}\epsilon^{2/3} k^{-5/3} \tag{61}$$

where K_{Ko} is a universal constant, commonly known as Kolmogorov's constant. Eq. (61) has been supported by numerous experiments and numerical simulations. Kolmogorov's constant K_{Ko} has been found to lie between 1.4-1.6 or so. It is quite amazing that complex interactions among fluid eddies in various different situations can be quite well approximated by Eq. (61).

Kolmogorov's derivation of Eq. (59) is quite involved. However, Eqs. (59, 61) can be derived using scaling arguments (dimensional analysis) under the assumption that

1. The energy spectrum in the inertial range does not depend on the large-scaling forcing processes and the small-scale dissipative processes, hence it must be a power law in the local wavenumber.

2. The energy transfer in fluid turbulence is local in the wavenumber space. The energy supplied to the fluid at the forcing scale cascades to smaller scales, and so on. Under steady-state the energy cascade rate is constant in the wavenumber space, i. e., $\Pi(k) = constant = \epsilon$.

In the framework of Kolmogorov's theory, several interesting deductions can be made.

1. Kolmogorov's theory assumes homogeneity and isotropy. In real flows, large-scales (forcing) as well as dissipative scales do not satisfy these properties. However, experiments and numerical simulations show that in the inertial range ($l_d \ll l \ll L$), the fluid flows are typically homogeneous and isotropic.

2. The velocity fluctuations at any scale l goes as

$$u_l \approx \epsilon^{1/3} l^{1/3}. \tag{62}$$

   Therefore, the effective time-scale for the interaction among eddies of size l is

$$\tau_l \approx \frac{l}{u_l} \approx \epsilon^{-1/3} l^{2/3}. \tag{63}$$

3. An extrapolation of Kolmogorov's scaling to the forcing and the dissipative scales yields

$$\epsilon \approx \frac{u_L^3}{L} \approx \frac{u_{l_d}^3}{l_d}. \tag{64}$$

   Taking $\nu \approx u_{l_d} l_d$, one gets

$$l_d \approx \left(\frac{\nu^3}{\epsilon}\right)^{1/4}. \tag{65}$$

   Note that the dissipation scale, also known as Kolmogorov's scale, depends on the large-scale quantity ϵ apart from kinematic viscosity.

4. From the definition of Reynolds number

$$Re = \frac{U_L L}{\nu} \approx \frac{U_L L}{u_{l_d} l_d} \approx \left(\frac{L}{l_d}\right)^{4/3} \tag{66}$$

   Therefore,

$$\frac{L}{l_d} \approx Re^{3/4}. \tag{67}$$

   Onset of turbulence depends on geometry, initial conditions, noise etc. Still, in most experiments turbulences sets in after Re of 2000 or more. Therefore, in three dimensions, number of active modes $(L/l_d)^3$ is larger than 26 million. These large number of modes make the problem quite complex and intractable.

5. Space dimension does not appear in the scaling arguments. Hence, one may expect Kolmogorov's scaling to hold in all dimensions. It is however found that the above scaling law is applicable in three dimension only. In two dimension (2D), conservation of enstrophy changes the behaviour significantly (see next two sections). The solution for one-dimensional incompressible Navier-Stokes is $\mathbf{u}(\mathbf{x}, t) = const$, which is a trivial solution.

6. Mode-to-mode energy transfer term $S(k|p|q)$ measures the strength of nonlinear interaction. Kolmogorov's theory implicitly assumes that energy cascades from larger to smaller scales. It is called local energy transfer in Fourier space. These issues will be discussed in Section VIII.

7. Careful experiments show that the spectral index is close to 1.71 instead of 1.67. This correction of ≈ 0.04 is universal and is due to the small-scale structures. This phenomena is known as intermittency, and will be discussed in Section IX.

8. Kolmogorov's model for turbulence works only for incompressible flow. It is connected to the fact that incompressible flow has local energy transfer in wavenumber space. Note that Burgers equation, which represents compressible flow ($U \gg C_s$), has k^{-2} energy spectrum, very different from Kolmogorov's spectrum.

Kolmogorov's theory of turbulence had a major impact on turbulence research because of its universality. Properties of scalar, MHD, Burgers, Electron MHD, wave turbulence have been studied using similar arguments.

As discussed in earlier sections, apart from energy spectra, there are many other quantities of interest in turbulence. Some of them are kinetic helicity, enstrophy etc. The statistical properties of these quantities are quite interesting, and they are addressed using Absolute Equilibrium State discussed below.

B. Absolute Equilibrium States

In fluid turbulence when viscosity is identically zero (inviscid limit), kinetic energy is conserved in the incompressible limit. Now consider independent Fourier modes (transverse to wavenumbers) as state variables $y_a(t)$. Lesieur [29] has shown that these variables move in a constant energy surface, and the motion is area preserving like in Liouville's theorem. Now we look for equilibrium probability-distribution function $P(\{y_a\})$ for these state variables. Once we assume ergodicity, the ideal incompressible fluid turbulence can be mapped to equilibrium statistical mechanics [29].

By applying the usual arguments of equilibrium statistical mechanics we can deduce that at equilibrium, the probability distribution function will be

$$P(y_1, ..., y_m) = \frac{1}{Z} \exp\left(-\frac{1}{2}\sigma \sum_{a=1}^{m} y_a^2\right),$$

where σ is a positive constant. The parameter σ corresponds to inverse temperature in the Boltzmann distribution. Clearly

$$\langle y_a^2 \rangle = \int \Pi_i dy_i y_a^2 P(\{y_i\}) = \frac{1}{\sigma},$$

independent of a. Hence energy spectrum $C(\mathbf{k})$ is constant, and 1-d spectrum will be proportional to k^{d-1} [29]. This is very different from Kolmogorov's spectrum for large Re turbulence. Hence, the physics of turbulence at $\nu = 0$ (inviscid) differs greatly from the physics at $\nu \to 0$. This is not surprising because (a) turbulence is a nonequilibrium process, and (b) Navier-Stokes equation is singular in ν.

Even though nature of inviscid flow is very different from turbulent flow, Kraichnan and Chen [26] suggested that the tendency of the energy cascade in turbulent flow could be anticipated from the absolute equilibrium states. Using absolute equilibrium theory, Kraichnan [24] showed that in two dimensions, enstrophy cascades forward, but energy cascades backward (see also Lesieur [29]). The above prediction holds good for real fluids.

V. EXPERIMENTAL RESULTS ON TURBULENCE

Analytical results are very rare in turbulence research because of complex nature of turbulence. Therefore, experiments and numerical simulations play very important role in turbulence research. In fluid turbulence, engineers have been able to obtain necessary information from experiments (e.g., wind tunnels), and successfully design complex machines like aeroplanes, spacecrafts etc. This aspect of fluid turbulence is not being covered here. For details on experiments, refer to books on fluid turbulence, e.g., Davidson [8].

VI. NUMERICAL INVESTIGATION OF FLUID TURBULENCE

Like experiments, numerical simulations help us test existing models and theories, and inspire new one. In addition, numerical simulations can be performed for conditions which may be impossible in real experiments, and all the field components can be probed everywhere, and at all times. Recent exponential growth in computing power has fueled major growth in this area of research. Of course, numerical simulations have limitations as well. Even the best computers of today cannot resolve all the scales in a turbulent flow. We will investigate these issues in this section.

There are many numerical methods to simulate turbulence on a computer. Engineers have devised many clever schemes to simulate flows in complex geometries; however, their attention is typically at large scales. Physicists normally focus on intermediate and small scales in a simple geometry because these scales obey universal laws. Since nonlinear equations are generally quite sensitive, one needs to compute both the spatial and temporal derivatives as accurately as possible. It has been shown that spatial derivative could be computed "exactly" using Fourier transforms given enough resolutions [4]. Therefore, physicists typically choose spectral method to simulate turbulence. Note however that several researchers have used higher order finite-difference scheme and have obtained comparable results.

A. Numerical Solution of fluid Equations using Pseudo-Spectral Method

In this subsection we will briefly sketch the spectral method for 3D flows. For details refer to Canuto et al. [4]. The fluid equations in Fourier space is written as

$$\frac{\partial \mathbf{u}(\mathbf{k},t)}{\partial t} = -i\mathbf{k}p(\mathbf{k},t) - FT\left[\mathbf{u}(\mathbf{k},t)\cdot\nabla\mathbf{u}(\mathbf{k},\mathbf{t})\right] - \nu k^2\mathbf{u}(\mathbf{k},t) + \mathbf{f}$$

where FT stands for Fourier transform, and $\mathbf{f}(\mathbf{k},t)$ is the forcing function. The flow is assumed to be incompressible, i. e., $\mathbf{k}\cdot\mathbf{u}(\mathbf{k},t)=0$. We assume periodic boundary condition with real-space box size as $(2\pi)\times(2\pi)\times(2\pi)$, and Fourier-space box size as (nx,ny,nz). The allowed wavenumbers are $\mathbf{k}=(k_x,k_y,k_z)$ with $k_x=(-n_x/2:n_x/2), k_y=(-n_y/2:n_y/2), k_z=(-n_z/2:n_z/2)$. The reality condition implies that $\mathbf{z}^{\pm}(-\mathbf{k})=\mathbf{z}^{\pm *}(\mathbf{k})$, therefore, we need to consider only half of the modes [4]. Typically we take $(-n_x/2:n_x/2,-n_y/2:n_y/2,0:n_z/2)$, hence, we have $N=n_x*n_y*(n_z/2+1)$ coupled ordinary differential equations. The objective is to solve for the field variables at a later time given initial conditions. The following important issues are involved in this method:

1. The Navier-Stokes equation is converted to nondimensionalized form, and then solved numerically. The parameter ν is inverse Reynold's number. Hence, for turbulent flows, ν is chosen to be quite small (typically 10^{-3} or 10^{-4}). In Section IV A we deduced using Kolmogorov's phenomenology that the number of active modes are

$$N \sim \nu^{-9/4}. \tag{68}$$

   If we choose a moderate Reynolds number $\nu^{-1}=10^4$, N will be 10^9, which is a very large number even for the most powerful supercomputers. To overcome this difficulty, researchers apply some tricks; the most popular among them are introduction of hyperviscosity and hyperresistivity, and large-eddy simulations. Hyperviscous (hyperresistive) terms are of the form $(\nu_j)k^{2j}\mathbf{u}(\mathbf{k})$ with $j\geq 2$; these terms become active only at large wavenumbers, and are expected not to affect the inertial range physics, which is of interest to us. Because of this property, the usage of hyperviscosity and hyperresistivity has become very popular in turbulence simulations. Large-eddy simulations are discussed in various books (e.g., see Pope [41]. Just to note, one of the highest resolution fluid turbulence simulation is by Gotoh [18] on a 4096^3 grid; this simulation was done on Fujitsu VPP5000/56 with 32 processors with 8 Gigabytes of RAM on each processor, and it took 500 hours of computer time.

2. The computation of the nonlinear terms is the most expensive part of turbulence simulation. A naive calculation involving convolution will take $O(N^2)$ floating point operations. It is instead efficiently computed using Fast Fourier Transform (FFT) as follows:
   (a) Compute $\mathbf{u}(\mathbf{x})$ from $\mathbf{u}(\mathbf{k})$ using Inverse FFT.
   (b) Compute $u_i(\mathbf{x})u_j(\mathbf{x})$ in real space by multiplying the fields at each space points.
   (c) Compute $FFT[u_i(\mathbf{x})u_j(\mathbf{x})]$ using FFT.
   (d) Compute $ik_jFFT[u_i(\mathbf{x})u_j(\mathbf{x})]$ by multiplying by k_j and summing over all j. This vector is $-FFT\left[\mathbf{u}(\mathbf{k},t)\cdot\nabla\mathbf{u}(\mathbf{k},\mathbf{t})\right]$.
   Since FFT takes $O(N\log N)$, the above method is quite efficient. The multiplication is done in real space, therefore this method is called pseudo-spectral method instead of just spectral method.

3. Products $u_i(\mathbf{x})u_j(\mathbf{x})$ produce modes with wavenumbers larger than k_{max}. On FFT, these modes get aliased with $k < k_{max}$ and will provide incorrect value for the convolution. To overcome this difficulty, last $1/3$ modes of fields $z_i^{\pm}(\mathbf{k})$ are set to zero (zero padding), and then FFTs are performed. This scheme is called 2/3 rule. For details refer to Canuto et al. [4].

4. Pressure is computed by taking the dot product of Navier-Stokes equation with $\mathbf{k}$. Using incompressibility condition one obtains

$$p\left(\mathbf{k},t\right) = \frac{i\mathbf{k}}{k^2} \cdot FT\left[\mathbf{u}\left(\mathbf{x},t\right) \cdot \nabla \mathbf{u}\left(\mathbf{x},\mathbf{t}\right)\right].$$

   To compute $p(\mathbf{k})$ we use already computed nonlinear term.

5. Once the right-hand side of the Navier-Stokes equation could be computed, we could time advance the equation using one of the standard techniques. The viscous terms are advanced using an implicit method called Crank-Nicholson's scheme. However, the nonlinear terms are advanced using Adam-Bashforth or Runge-Kutta scheme. One uses either second or third order scheme. Choice of dt is determined by CFL criteria ($dt < (\triangle x)/U_{rms}$). By repeated application of time-advancing, we can reach the desired final time.

6. When forcing $\mathbf{f} = 0$, the total energy gets dissipated due to viscosity. This is called decaying simulation. On the contrary, forced simulation have nonzero forcing ($\mathbf{f} \neq 0$), which feed energy into the system, and the system typically reaches a steady-state in several eddy turnover time. Forcing in turbulent systems are typically at large-scale eddies (shaking, stirring etc.). Therefore, in forced turbulence $\mathbf{f}$ is typically applied at small wavenumbers, which could feed kinetic energy and kinetic helicity.

Spectral method has several disadvantages as well. This method can not be easily applied to nonperiodic flows. That is the reason why engineers hardly use spectral method. Note however that even in aperiodic flows with complex boundaries, the flows at small length-scale can be quite homogeneous, and can be simulated using spectral method. Spectral simulations are very popular among physicists who try to probe universal features of small-scale turbulent flows.

The numerical results on turbulent energy spectrum and fluxes are described in many turbulence literature, e. g., Lesieur [29]. In Section 8 we describe a numerical result on shell-to-shell energy transfer in 512^3 simulation. In recent times a technique called large-eddy simulation (LES) has become very popular. LES enables us to perform turbulence simulations on smaller grids. In this paper we donot cover this topic.

In the next three sections we will describe the field-theoretic calculation of renormalized viscosity.

VII. RENORMALIZATION GROUP ANALYSIS OF FLUID TURBULENCE

In Section ?? we discussed various existing turbulence models. Here we will describe some field-theoretic calculations.

Field theory is well developed, and has been applied to many areas of physics, e.g., Quantum Electrodynamics, Condensed Matter Physics etc. In this theory, the equations are expanded perturbatively in terms of nonlinear term, which are considered small. In fluid turbulence the nonlinear term is not small; the ratio of nonlinear to linear (viscous) term is Reynolds numbers, which is large in turbulence regime. This problem appears in many areas of physics including Quantum Chromodynamics (QCD), Strongly Correlated Systems, Quantum Gravity etc., and is largely unsolved. To overcome the above difficulty, some clever schemes have been adopted such as Direct Interaction Approximation, Renormalization Groups (RG), Eddy-damped quasi-normal Markovian approximations, etc. We discuss some of them below. A simple-minded calculation of Green's function shows divergence at small wavenumbers (infrared divergence). One way to solve this problem is by introducing an infrared cutoff for the integral. The reader is referred to Leslie [30] for details. RG technique, to be described below, is a systematic procedure to cure this problem.

A. Renormalization Groups in Turbulence

Renormalization Group Theory (RG) is a technique which is applied to complex problems involving many length scales. Many researchers have applied RG to turbulence. Over the years, several different RG applications for turbulence has been discovered. Broadly speaking, they fall in three different categories:

Yakhot-Orszag (YO) Perturbative approach

Yakhot and Orszag's [58] work, motivated by Forster *et al.* [12] and Fournier and Frisch [14], is the first comprehensive application of RG to turbulence. It is based on Wilson's shell-elimination procedure. Also refer to Smith and Woodruff [44] for details. Here the renormalized parameter is function of forcing noise spectrum $D(k) = D_0 k^{-y}$. It is shown that the local Reynolds number $\bar{\lambda}$ is

$$\bar{\lambda} = \frac{\lambda_0^2 D_0}{\nu^3(\Lambda)\Lambda^\epsilon},$$

where λ_0 is the expansion parameter, Λ is the cutoff wavenumber, and $\epsilon = 4+y-d$ [58]. It is found that $\nu(\Lambda)$ increases as Λ decreases, therefore, $\bar{\lambda}$ remains small (may not be less that one though) compared to Re as the wavenumber shells are eliminated. Hence, the "effective" expansion parameter is small even when the Reynolds number may be large.

The RG analysis of Yakhot and Orszag [58] yielded Kolmogorov's constant $K_{Ko} = 1.617$, turbulent Prandtl number for high-Reynolds-number heat transfer $P_t = 0.7179$, Batchelor constant $Ba = 1.161$ etc. These numbers are quite close to the experimental results. Hence, Yakhot and Orszag's method appears to be highly successful. However there are several criticisms to the YO scheme. Kolmogorov's spectrum results in the YO scheme for $\epsilon = 4$, far away from $\epsilon = 0$, hence epsilon-expansion is questionable. YO proposed that higher order nonlinearities are "irrelevant" in the RG sense for $\epsilon = 0$, and are marginal when $\epsilon = 4$. Eyink [11] objected to this claim and demonstrated that the higher order nonlinearities are marginal regardless of ϵ. Kraichnan [25] compared YO's procedure with Kraichnan's Direct Interaction Approximation [23] and raised certain objections regarding distant-interaction in YO scheme. For details refer to Zhou and McComb [63] and Smith and Woodruff [44].

Self-consistent approach of McComb and Zhou

This is one of the nonperturbative method, which is often used in Quantum Field theory. In this method, a self-consistent equation of the full propagator is written in terms of itself and the proper vertex part. The equation may contain many (possibly infinite) terms, but it is truncated at some order. Then the equation is solved iteratively. McComb [33], Zhou and coworkers [65] have applied this scheme to fluid turbulence, and have calculated renormalized viscosity and Kolmogorov's constant successfully. Direct Interaction Approximation of Kraichnan is quite similar to self-consistent theory (Smith and Woodruff [44]).

The difficulty with this method is that it is not rigorous. In McComb and Zhou's procedures, the vertex correction is not taken into account. Verma [49, 50, 51] has applied the self-consistent theory to MHD turbulence.

Figure 3: The wavenumber shells to be averaged during renormalization procedure.

Callan-Symanzik Equation for Turbulence

DeDominicis and Martin [9] and Teodorovich [47] obtained the RG equation using functional integral. Teodorovich obtained $K_{Ko} = 2.447$, which is in not in good agreement with the experimental data, though it is not too far away. It has been shown that Wilson's shell-renormalization and RG through Callan-Symanzik equation are equivalent procedure. However, careful comparison of RG schemes in turbulence is not completely worked out.

In the following discussion we will discuss McComb's RG scheme is some detail. The other schemes have been discussed in great lengths in several books and review articles. After renormalization, in Section VIII we will discuss the computation of energy fluxes. These calculations are done using self-consistent field theory, a scheme very similar to DIA. At the end we will describe Eddy-damped quasi-normal Markovian approximation, which is very similar to the energy flux calculation.

B. Physical Meaning of Renormalization in Turbulence

The field theorists have been using renormalization techniques since 1940s. However, the physical meaning of renormalization became clear after path-breaking work of Wilson [57]. Here renormalization is a variation of parameters as we go from one length scale to the next. Following Wilson, renormalized viscosity and resistivity can also be interpreted as scale-dependent parameters. We coarse-grain the physical space and look for an effective theory at a larger scale. In this method, we sum up all the interactions at smaller scales, and as a outcome we obtain terms that can be treated as a correction to viscosity and resistivity. The corrected viscosity and resistivity are called "effective" or renormalized dissipative parameters. This procedure of coarse graining is also called shell elimination in wavenumber space. We carry on with this averaging process till we reach inertial range. In the inertial range the "effective" or renormalized parameters follow a universal powerlaw, e. g., renormalized viscosity $\nu(l) \propto l^{4/3}$. This is the renormalization procedure in turbulence. Note that the renormalized parameters are independent of microscopic viscosity or resistivity.

In viscosity renormalization the large wavenumber shells are eliminated, and the interaction involving these shells are summed. Hence, we move from larger wavenumbers to smaller wavenumbers. However, it is also possible to go from smaller wavenumbers to larger wavenumber by summing the smaller wavenumber shells, e.g, for shear flows. This process is not coarse-graining, but it is a perfectly valid RG procedure, and is useful when the small wavenumber modes (large length scales) are linear. This scheme is followed in Quantum Electrodynamics (QED), where the electromagnetic field is negligible at a large distance (small wavenumbers) from a charge particle, while the field becomes nonzero at short distances (large wavenumber). In QED, the charge of a particle gets renormalized when we come closer to the charge particle, i. e., from smaller wavenumbers to larger wavenumbers. See Fig. 3 for an illustration of wavenumber shells to be averaged. In the following subsection we will calculate renormalized viscosity using RG procedure.

C. Renormalization of viscosity using self-consistent procedure

In this subsection we compute renormalized viscosity using self-consistent procedure. This work was done by McComb and his group workers. The renormalization of viscosity is performed *from large wavenumber to smaller wavenumbers.*

McComb and his group workers took the following form of Kolmogorov's spectrum for kinetic energy

$$E(k) \;=\; K_{Ko}\Pi^{2/3}k^{-5/3}, \tag{69}$$

where K_{Ko} is Kolmogorov's constant, and Π is the total energy flux. The incompressible fluid equations in the Fourier space are

$$\left(-i\omega+\nu k^2\right)u_i\left(\hat{k}\right) \;=\; -\frac{i}{2}P^{+}_{ijm}(\mathbf{k})\int_{\hat{p}+\hat{q}=\hat{k}} d\hat{p}\left[u_j(\hat{p})u_m(\hat{q})\right], \tag{70}$$

where

$$P^{+}_{ijm}(\mathbf{k}) \;=\; k_jP_{im}(\mathbf{k})+k_mP_{ij}(\mathbf{k}), \tag{71}$$

Here ν is viscosity, and d is the space dimensionality.

In this RG procedure the wavenumber range (k_N, k_0) is divided logarithmically into N shells. The nth shell is (k_n, k_{n-1}) where $k_n = h^n k_0$ $(h<1)$. In the following discussion, the elimination of the first shell (k_1, k_0) is carried out, and modified NS equation is obtained. Then one proceeds iteratively to eliminate higher shells and get a general expression for the modified fluid equation. The renormalization group procedure is as follows:

1. The the spectral space is divided in two parts: 1. the shell $(k_1,k_0) = k^{>}$, which is to be eliminated; 2. $(k_N,k_1) = k^{<}$, set of modes to be retained. Note that $\nu_{(0)}$ denote the viscosity and resistivity before the elimination of the first shell.

2. Rewrite Eqs. (70) for $k^{<}$ and $k^{>}$. The equations for $u_i^{<}(\hat{k})$ and $b_i^{<}(\hat{k})$ modes are

$$\begin{aligned}\left(-i\omega+\Sigma_{(0)}(k)\right)u_i^{<}(\hat{k}) \;=\; &-\frac{i}{2}P^{+}_{ijm}(\mathbf{k})\int d\hat{p}([u_j^{<}(\hat{p})u_m^{<}(\hat{k}-\hat{p})]\\ &+2[u_j^{<}(\hat{p})u_m^{>}(\hat{k}-\hat{p})]+[u_j^{>}(\hat{p})u_m^{>}(\hat{k}-\hat{p})])\end{aligned} \tag{72}$$

   The Σs appearing in the equations are usually called the "self-energy" in Quantum field theory language. In the first iteration, $\Sigma_{(0)} = \nu_{(0)}k^2$. The equation for $u_i^{>}(\hat{k})$ modes can be obtained by interchanging $<$ and $>$ in the above equations.

3. The terms given in the second and third brackets in the Right-hand side of Eqs. (72) are calculated perturbatively. Since we are interested in the statistical properties of $\mathbf{u}$ fluctuations, we perform the usual ensemble average of the system [58]. It is assumed that $\mathbf{u}^{>}(\hat{k})$ has gaussian distributions with zero mean, while $\mathbf{u}(\hat{k})$ is unaffected by the averaging process. Hence,

$$\left\langle u_i^{>}(\hat{k})\right\rangle \;=\; 0 \tag{73}$$

$$\left\langle u_i^{<}(\hat{k})\right\rangle \;=\; u_i^{<}(\hat{k}) \tag{74}$$

   and

$$\left\langle u_i^{>}(\hat{p})u_j^{>}(\hat{q})\right\rangle \;=\; P_{ij}(\mathbf{p})C(\hat{p})\delta(\hat{p}+\hat{q}) \tag{75}$$

   The triple order correlations $\left\langle u_i^{>}(\hat{k})u_j^{>}(\hat{p})u_m^{>}(\hat{q})\right\rangle$ are zero due to Gaussian nature of the fluctuations. Here, X stands for u or b. In addition, we also neglect the contribution from the triple nonlinearity $\left\langle u^{<}(\hat{k})u_j^{<}(\hat{p})u_m^{<}(\hat{q})\right\rangle$, as done in many of the turbulence RG calculations [33, 58]. The effects of triple nonlinearity can be included following the scheme of Zhou and Vahala [65].

4. To the first order, the second bracketed terms of Eqs. (72) vanish, but the nonvanishing third bracketed terms yield corrections to Σs. Refer to Appendix C for details. Eqs. (72) can now be approximated by

$$\left(-i\omega + \Sigma_{(0)} + \delta\Sigma_{(0)}\right) u_i^<(\hat{k}) \;=\; -\frac{i}{2} P^+_{ijm}(\mathbf{k}) \int d\hat{p}[u_j^<(\hat{p}) u_m^<(\hat{k}-\hat{p})] \tag{76}$$

with

$$\delta\Sigma^{uu}_{(0)}(k) \;=\; \frac{1}{(d-1)} \int_{\hat{p}+\hat{q}=\hat{k}}^{\Delta} d\hat{p}[S(k,p,q) G(\hat{p}) C(\hat{q})] \tag{77}$$

where

$$S(k,p,q) = kp((d-3)z + 2z^3 + (d-1)xy). \tag{78}$$

The integral Δ is to be done over the first shell.

5. The frequency dependence of the correlation function is taken as: $C(k,\omega) = 2C(k)\Re(G(k,\omega))$. In other words, the relaxation time-scale of correlation function is assumed to be the same as that of corresponding Green's function. Since we are interested in the large time-scale behaviour of turbulence, we take the limit ω going to zero. Under these assumptions, the frequency integration of the above equations yield

$$\delta\nu_{(0)}(k) \;=\; \frac{1}{(d-1)k^2} \int_{\mathbf{p}+\mathbf{q}=\mathbf{k}}^{\Delta} \frac{d\mathbf{p}}{(2\pi)^d} \frac{S(k,p,q)C(q)}{\nu_{(0)}(p)p^2 + \nu_{(0)}(q)q^2} \tag{79}$$

Note that $\nu(k) = \Sigma^{uu}(k)/k^2$. There are some important points to remember in the above step. The frequency integral in the above is done using contour integral. It can be shown that the integrals are nonzero only when both the components appearing the denominator are of the same sign. For example, first term of Eq. (79) is nonzero only when both $\nu_{(0)}(p)$ and $\nu_{(0)}(q)$ are of the same sign.

6. Let us denote $\nu_{(1)}(k)$ as the renormalized viscosity after the first step of wavenumber elimination. Hence,

$$\nu_{(1)}(k) \;=\; \nu_{(0)}(k) + \delta\nu_{(0)}(k); \tag{80}$$

We keep eliminating the shells one after the other by the above procedure. After $n+1$ iterations we obtain

$$\nu_{(n+1)}(k) = \nu_{(n)}(k) + \delta\nu_{(n)}(k) \tag{81}$$

where the equation for $\delta\nu_{(n)}(k)$ is the same as the Eqs. (79) except that $\nu_{(0)}(k)$ appearing in the equation is to be replaced by $\nu_{(n)}(k)$. Clearly $\nu_{(n+1)}(k)$ is the renormalized viscosity and resistivity after the elimination of the $(n+1)$th shell.

7. We need to compute $\delta\nu_{(n)}$ for various n. These computations, however, require $\nu_{(n)}$. In our scheme we solve these equations iteratively. In Eqs. (79, **??**) we substitute $C(k)$ by one dimensional energy spectrum $E(k)$

$$C(k) = \frac{2(2\pi)^d}{S_d(d-1)} k^{-(d-1)} E(k)$$

where S_d is the surface area of d-dimensional spheres. We assume that $E(k)$ follows Eqs. (69) . Regarding $\nu_{(n)}$, we attempt the following form of solution

$$\nu_{(n)}(k_n k') = (K_{Ko})^{1/2} \Pi^{1/3} k_n^{-4/3} \nu^*_{(n)}(k')$$

with $k = k_{n+1}k' (k' < 1)$. We expect $\nu^*_{(n)}(k')$ to be a universal functions for large n. The substitution of $C(k), \nu_{(n)}(k)$ yields the following equations:

$$\delta\nu^*_{(n)}(k') \;=\; \frac{1}{(d-1)} \int_{\mathbf{p'}+\mathbf{q'}=\mathbf{k'}} d\mathbf{q}' \frac{2}{(d-1)S_d} \frac{E^u(q')}{q'^{d-1}} [S(k',p',q') \frac{1}{\nu^*_{(n)}(hp')p'^2 + \nu^*_{(n)}(hq')q'^2}] \tag{82}$$

$$\nu^*_{(n+1)}(k') \;=\; h^{4/3}\nu^*_{(n)}(hk') + h^{-4/3}\delta\nu^*_{(n)}(k') \tag{83}$$

where the integrals in the above equations are performed iteratively over a region $1 \le p', q' \le 1/h$ with the constraint that $\mathbf{p}' + \mathbf{q}' = \mathbf{k}'$. Fournier and Frisch [13] showed the above volume integral in d dimension to be

$$\int_{\mathbf{p}'+\mathbf{q}'=\mathbf{k}'} d\mathbf{p}' = S_{d-1} \int dp' dq' \left(\frac{p'q'}{k'}\right)^{d-2} (\sin\alpha)^{d-3}, \tag{84}$$

where α is the angle between vectors $\mathbf{p}'$ and $\mathbf{q}'$.

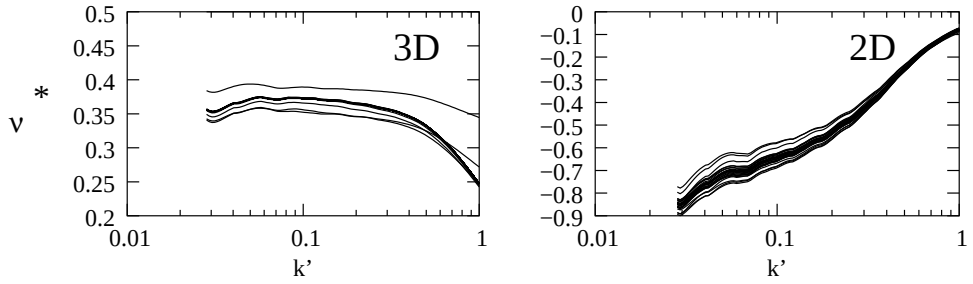


Figure 4: Plot of $\nu^*(k')$ vs. k' for 2D and 3D fluid turbulence. In 2D, ν^* is negative.

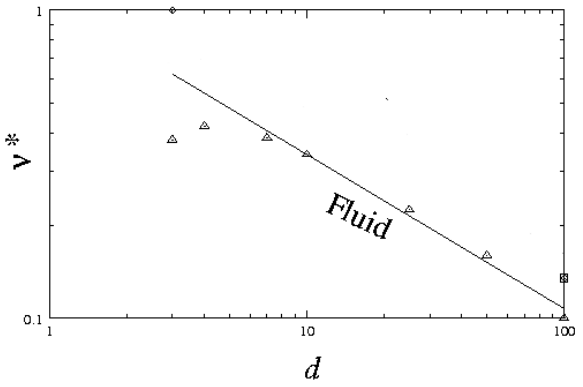


Figure 5: The plot of asymptotic ν^*vs. d . For large d, the plot fits quite well with predicted $d^{-1/2}$ curve. Adopted from Verma [50].

8. Now the above equations are solved self-consistently with $h = 0.7$. This value is about middle of the range (0.55-0.75) estimated to be the reasonable values of h by Zhou *et al.* [63]. One starts with constant value of $\nu^*_{(0)}$, and compute the integrals using Gauss quadrature technique. Once $\delta\nu^*_{(0)}$ has been computed, $\nu^*_{(1)}$ is computed. This process is iterated till $\nu^*_{(m+1)}(k') \approx \nu^*_{(m)}(k')$, that is, till they converge. The result of our RG analysis is given below.

McComb and coworkers [33, 36, 63] successfully applied the above self-consistent renormalization group theory to 2D and 3D fluid turbulence. They found that $\nu^*(k')$ converges quite quickly. For 3D the value of $\nu^*(k' \to 0)$ is approximately 0.38. See Fig. 4 for an illustration.

For 2D turbulence $\nu^*(k')$ is negative as shown in Fig. 4. The function ν^* is not very well behaved as $k' \to 0$. Still, negative renormalized viscosity is consistent with negative eddy viscosity obtained using Test Field Model [24] and EDQNM calculations [42]. We estimate $\nu^* \approx -0.60$.

For large d , $\nu^* = \eta^*$, and it decreases as $d^{-1/2}$ (see Fig. 5); ν^* for pure fluid turbulence also decreases as $d^{-1/2}$, as

shown in the same figure. This is evident from Eqs. (82) [15]. For large d

$$\int dp'dq' \left(\frac{p'q'}{k'}\right)^{d-2} (\sin\alpha)^{d-3} ... \sim d^{-1/2}, \tag{85}$$

$$\frac{S_{d-1}}{(d-1)^2 S_d} \sim \frac{1}{d^2}\left(\frac{d}{2\pi}\right)^{1/2},$$

$$S, -S_6, -S_8, S_9(k', p', q') = kpd(z+xy), \tag{86}$$

which leads to

$$\nu^* \delta\nu^* \propto \frac{1}{d^2}\left(\frac{d}{2\pi}\right)^{1/2} d^{-1/2} d$$

hence $\nu^* \propto d^{-1/2}$.

In conclusion the above RG procedure shows that

$$E(k) = K_{Ko}\Pi^{2/3}k^{-5/3}, \tag{87}$$

$$\nu(k = k_n k') = K_{Ko}^{1/2}\Pi^{1/3}k_n^{-4/3}\nu^*(k'), \tag{88}$$

is a consistent solution of renormalization group equation. Here, K_{Ko} is Kolmogorov's constant, Π is the energy flux, and $\nu^*(k')$ is a universal function that is a constant as $k' \to 0$.

1. *Helical turbulence*

Helical turbulence is defined for space dimension $d = 3$. We can extend the above the RG analysis to helical turbulence (Zhou [61]). All the steps are the same except Eqs. (75) are replaced by

$$\left\langle u_i^>(\hat{p}) u_j^>(\hat{q})\right\rangle = \left[P_{ij}(\mathbf{p})C^{uu}(\hat{p}) - i\epsilon_{ijl}\frac{p_l}{p^2}H_K(\hat{p})\right](2\pi)^4\delta(\hat{p}+\hat{q}) \tag{89}$$

Because of helicities, the equation for change in renormalized self-energy (79) gets altered to

$$\delta\nu_{(0)}(k) = \frac{1}{(d-1)k^2}\int_{\mathbf{p}+\mathbf{q}=\mathbf{k}}^{\Delta}\frac{d\mathbf{p}}{(2\pi)^d}[\frac{S(k,p,q)C^{uu}(q) + S'(k,p,q)H_K(q)}{\nu_{(0)}(p)p^2 + \nu_{(0)}(q)q^2}$$

where S_i' defined below can be shown to be zero.

$$S'(k,p,q) = P^+_{bjm}(k)P^+_{mab}(p)\epsilon_{jal}q_l = 0,$$

The argument for vanishing of S' is follows. Since $\delta\nu$ is a proper scalar, and H_K is a pseudo scalar, $S'(k,p,q)$ will be also be a pseudo scalar. In addition, $S'(k,p,q)$ are also linear in k, p and q. This implies that $S_i'(k,p,q)$ must be proportional to $\mathbf{q}\cdot(\mathbf{k}\times\mathbf{p})$, which will be zero because $\mathbf{k} = \mathbf{p}+\mathbf{q}$. Hence $S'(k,p,q)$ turn out to be zero. *Hence, helicities do not alter the already calculated $\delta(\nu)_{(n)}(k)$ in the earlier section.*

In fluid turbulence, there are some other interesting variations of field-theoretic calculations by DeDominicis and Martin [9], Bhattacharjee [2], Carati [5] and others.

In the next section we will compute energy fluxes for fluid turbulence using field-theoretic techniques.

VIII. FIELD-THEORETIC CALCULATION OF ENERGY FLUXES AND SHELL-TO-SHELL ENERGY TRANSFER

In this section we present calculation of energy flux using field-theoretic method. We assume the turbulence to be homogeneous and isotropic. Even though the real-world turbulence do not satisfy these properties, many conclusions drawn using these assumption provide us with important insights into the energy transfer mechanisms at small scales. The field-theoretic procedure requires Fourier space integrations of functions involving products of energy spectrum and the Greens functions. Since there is a general agreement on Kolmogorov-like spectrum for fluid turbulence, $E(k) \propto k^{-5/3}$ is taken for the energy spectrum. For the Greens function, we substitute the "renormalized" or "dressed" Greens function computed in the previous section [50] (see Section VII C).

A. Calculation of Energy Flux

As described in Section III the energy flux from a wavenumber sphere of radius k_0 to the outside of the sphere of the same radius is

$$\Pi(k_0) = \frac{1}{(2\pi)^d \delta(\mathbf{k}'+\mathbf{p}+\mathbf{q})} \int_{k'>k_0} \frac{d\mathbf{k}'}{(2\pi)^d} \int_{p<k_0} \frac{d\mathbf{p}}{(2\pi)^d} \langle S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})\rangle \tag{90}$$

We assume that the kinetic energy is forced at small wavenumbers.

We analytically calculate the above energy fluxes in the inertial range to leading order in perturbation series. It was assumed that $\mathbf{u}(\mathbf{k})$ is quasi-gaussian as in EDQNM approximation. Under this approximation, the triple correlation $\langle XXX\rangle$ is zero to zeroth order, but nonzero to first oder. To first order $\langle XXX\rangle$ is written in terms of $\langle XXXX\rangle$, which is replaced by its Gaussian value, a sum of products of second-order moment. Consequently, the ensemble average of S, $\langle S\rangle$, is zero to the zeroth order, but is nonzero to the first order. The first order terms for $\langle S(k|p|q)\rangle$ in terms of Feynman diagrams are given in Appendix C. They are given below in terms of Green's functions and correlation functions:

$$\begin{aligned}\langle S(k|p|q)\rangle = \int_{-\infty}^{t} dt'(2\pi)^d [&T_1(k,p,q)G(k,t-t')C(p,t,t')C(q,t,t') \\ &+T_2(k,p,q)G(p,t-t')C(k,t,t')C(q,t,t') \\ &+T_3(k,p,q)G(q,t-t')C(k,t,t')C(p,t,t')]\delta(\mathbf{k}'+\mathbf{p}+\mathbf{q})\end{aligned} \tag{91}$$

where $T_i(k,p,q)$ are functions of wavevectors k, p, and q given in Appendix B.

The Greens functions can be written in terms of "effective" or "renormalized" viscosity $\nu(k)$ and resistivity $\eta(k)$ computed in Section VII

$$G(k,t-t') = \theta(t-t')\exp\left(-\nu(k)k^2(t-t')\right)$$

The relaxation time for $C(k,t,t')$ is assumed to be the same as that of $G(k,t,t')$. Therefore the time dependence of the unequal-time correlation functions will be

$$C^{uu}(k,t,t') = \theta(t-t')\exp\left(-\nu(k)k^2(t-t')\right) C^{uu,bb}(k,t,t)$$

The above forms of Green's and correlation functions are substituted in the expression of $\langle S\rangle$, and the t' integral is performed. Now Eq. (90) yields the following flux formula for $\Pi(k_0)$:

$$\begin{aligned}\Pi(k_0) = \int_{k>k_0} \frac{d\mathbf{k}}{(2\pi)^d} \int_{p<k_0} \frac{d\mathbf{k}}{(2\pi)^d} &\frac{1}{\nu(k)k^2+\nu(p)p^2+\nu(q)q^2} \times [T_1(k,p,q)C^{uu}(p)C^{uu}(q) \\ &+T_2(k,p,q)C^{uu}(k)C^{uu}(q)+T_3(k,p,q)C^{uu}(k)C^{uu}(p)].\end{aligned} \tag{92}$$

The expressions for the other fluxes can be obtained similarly.

The equal-time correlation function $C(k,t,t)$ at the steady-state can be written in terms of one dimensional energy spectrum as

$$C(k,t,t) = \frac{2(2\pi)^d}{S_d(d-1)} k^{-(d-1)} E(k),$$

where S_d is the surface area of d-dimensional unit spheres. We are interested in the fluxes in the inertial range. Therefore, we substitute Kolmogorov's spectrum [Eqs.(69)] for the energy spectrum. The effective viscosity is proportional to $k^{-4/3}$, i.e.,

$$\nu(k) = (K^u)^{1/2}\Pi^{1/3}k^{-4/3}\nu^*, \tag{93}$$

and the parameter ν^* was calculated in Section VII.

We nondimensionalize Eq. (92) by substituting [30]

$$k = \frac{k_0}{u}; \quad p = \frac{k_0}{u}v; \quad q = \frac{k_0}{u}w. \tag{94}$$

Application of Eq. (84) yields

$$\Pi = (K_{Ko})^{3/2}\Pi\left[\frac{4S_{d-1}}{(d-1)^2 S_d}\int_0^1 dv \ln(1/v)\int_{1-v}^{1+v} dw(vw)^{d-2}(\sin\alpha)^{d-3}F(v,w)\right], \tag{95}$$

where the integral $F(v,w)$ are

$$F = \frac{1}{\nu^*(1+v^{2/3}+w^{2/3})}[t_1(v,w)(vw)^{-d-\frac{2}{3}} + t_2(v,w)w^{-d-\frac{2}{3}} + t_3(v,w)v^{-d-\frac{2}{3}}], \tag{96}$$

Here $t_i(v,w) = T_i(k,kv,kw)/k^2$. Note that the energy fluxes are constant, consistent with the Kolmogorov's picture. We compute the bracketed term (denoted by I) numerically using Gaussian-quadrature method, and found it to be convergent. Using I the constant K_{Ko} can be calculated as

$$K_{Ko} = (I)^{-2/3}. \tag{97}$$

For 3D turbulence, the value of constant K_{Ko} computed using Eqs. (95, 97) is 1.58. This number is very good agreement with numerical and experimental estimate of Kolmogorov's constant. For 2D turbulence, we substitute $\nu^* = -0.60$ in the above equations. The computation yields $K_{Ko}^{2D} \approx 6.3$.

For large d

$$\int dp'dq'\left(\frac{p'q'}{k'}\right)^{d-2}(\sin\alpha)^{d-3}... \sim d^{-1/2}, \tag{98}$$

$$\frac{S_{d-1}}{(d-1)^2 S_d} \sim \frac{1}{d^2}\left(\frac{d}{2\pi}\right)^{1/2},$$

$$\nu^* = \eta^* \sim d^{-1/2} \tag{99}$$

$$t_1 = -t_2 = kpd(z+xy), \tag{100}$$

and $t_3 = 0$. Using Eq. (100) and by matching the dimensions, it can be shown that $K \propto d^{-1/3}$. This result is due to Fournier *et al.* [13].

All the above conclusions are for large Reynolds number or $\nu \to 0$ limit. The behaviour of Navier-Stokes equation for viscosity $\nu = 0$ (inviscid) is very different, and has been analyzed using absolute equilibrium theory (see Section IV B). It can be shown using this theory that under steady state, energy is equipartitioned among all the modes, hence $C(k) = const$ [39]. Using this result we can compute mode-to-mode energy transfer rates $\langle S^{uu}(k|p|q)\rangle$ to first order in perturbation theory (Eq. [91]), which yields

$$\langle S^{uu}(k|p|q)\rangle \propto \int \frac{(T_1(k,p,q) + T_5(k,p,q) + T_9(k,p,q))\,Const}{\nu(k)k^2 + \nu(p)p^2 + \nu(q)q^2} = 0$$

because $T_1(k,p,q) + T_5(k,p,q) + T_9(k,p,q) = 0$. Hence, under steady-state, their is no energy transfer among Fourier modes in inviscid Navier-Stokes. In other words "principle of detailed balance" holds here. Note that the above result holds for all space dimensions. Contrast this result with the turbulence situation when energy preferentially gets transferred from smaller wavenumber to larger wavenumber. This example contrasts equilibrium and nonequilibrium systems.

After completing the discussion on energy fluxes for fluid turbulence, we now move on to theoretical computation of shell-to-shell energy transfer.

B. Field-theoretic Calculation of Shell-to-shell Energy Transfer

Energy transfers between wavenumber shells provide us with important insights into the dynamics of turbulence. Kolmogorov's fluid turbulence model is based on local energy transfer between wavenumber shells. There are several quantitative theories in fluid turbulence about the amount of energy transfer between neighbouring wavenumber shells. For examples, Kraichnan [24] showed that 35% of the energy transfer comes from wavenumber triads where the smallest wave-number is greater than one-half of the middle wavenumber.

In this subsection we will compute the shell-to-shell energy transfer in turbulence using field-theoretic method [55]. The procedure is identical to the one described for energy fluxes. Recall that the energy transfer rates from the m-th shell to the n-th shell is

$$T_{nm} = \sum_{\mathbf{k}'\in n} \sum_{\mathbf{p}\in m} S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}). \tag{101}$$

The $\mathbf{p}$-sum is over m-th shell, and the $\mathbf{k}'$-sum is over n-th shell (Section III). The terms of S's are the same as in flux calculation, however, the limits of the integrals are different. The shells are binned logarithmically with n-th shell being $(k_0 s^{n-1}, k_0 s^n)$. We nondimensionalize the equations using the transformation [30]

$$k = \frac{a}{u}; \qquad p = \frac{a}{u}v; \qquad q = \frac{a}{u}w, \tag{102}$$

where $a = k_0 s^{n-1}$. The resulting equation is

$$\frac{T_{nm}}{\Pi} = K_{Ko}^{3/2} \frac{4S_{d-1}}{(d-1)^2 S_d} \int_{s^{-1}}^{1} \frac{du}{u} \int_{us^{m-n}}^{us^{m-n+1}} dv \int_{|1-v|}^{1+v} dw\, (vw)^{d-2} (\sin\alpha)^{d-3} F(v,w), \tag{103}$$

where$F(v,w)$ was computed in the previous section. The renormalized parameters ν^*, and Kolmogorov's constant K_{Ko} required to compute T_{nm}/Π are taken from the previous calculations. From Eq. (103) we can draw the following inferences:

1. The shell-to-shell energy transfer rate is a function of $n-m$, that is, $\Phi_{nm} = \Phi_{(n-i)(m-i)}$. Hence, the turbulent energy transfer rates in the inertial range are all self-similar. Of course, this is true only in the inertial range.
2. $T_{nn}/\Pi = 0$.

We compute the integral of Eq. (103) and substitute the value of K_{Ko}, which yields T_{nm}. The plots of $T_{nm}$for 2D and 3D fluids are shown in Fig. 6. In 3D the energy transfer is forward and local. In 2D however the energy transfer is forward for the nearest neighbours, but it is backward for fourth neighbour onward; these backward transfers are one of the major factors in the inverse cascade of energy [56]. The sum of all these transfers is negative energy flux, consistent with the inverse cascade result of Kraichnan [24]. For details refer to Verma et al. [56].

Verma et al. [56] computed the shell-to-shell energy transfer in 3D fluid turbulence using numerical simulations. Their result is shown in Fig. 7. Comparison of Fig. 6 and Fig. 7 shows that theoretical and numerical computation of shell-to-shell energy transfer are consistent with each other.

Incompressible fluid turbulence is nonlocal in real space due to incompressibility condition. Field-theoretic calculation also reveals that mode-to-mode transfer $S(k|p|q)$ is large when $p \ll k$, but small for $k \sim p \sim q$, hence Navier-Stokes equation is nonlocal in Fourier space too. However, in 3D shell-to-shell energy transfer rate T_{nm} is forward and most significant to the next-neighbouring shell. Hence, shell-to-shell energy transfer rate is local even though the interactions appear to be nonlocal in both real and Fourier space. Refer to Zhou [62], Domaradzki and Rogallo [10], Verma et al. [56], and Verma [54].

With this we conclude our discussion on shell-to-shell energy transfer in fluid transfer.

C. EDQNM Calculation of Fluid Turbulence

Eddy-damped quasi-normal Markovian (EDQNM) calculation of turbulence is very similar to the field-theoretic calculation of energy evolution. This scheme was first invented by Orszag [39] for Fluid turbulence.

The Navier-Stokes equation is symbolically written as

$$\left(\frac{d}{dt} + \zeta k^2\right) X(\mathbf{k},t) = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} X(\mathbf{p},t) X(\mathbf{q},t),$$

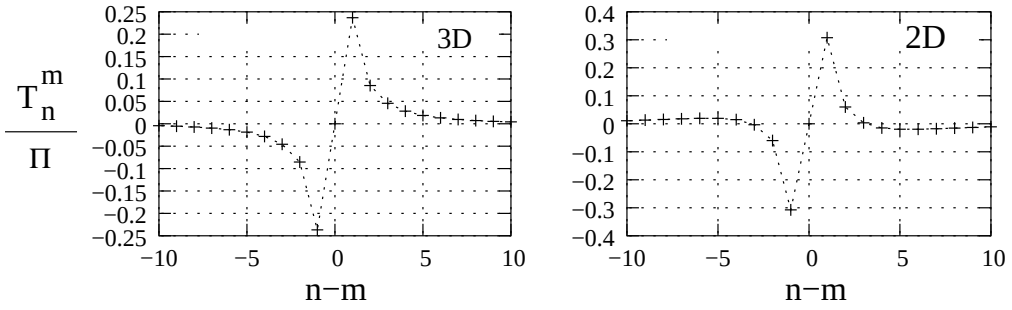


Figure 6: Plots of shell-to-shell energy transfer rates T_{nm}/Π vs. $n-m$ for 3D and 2D fluid turbulence.

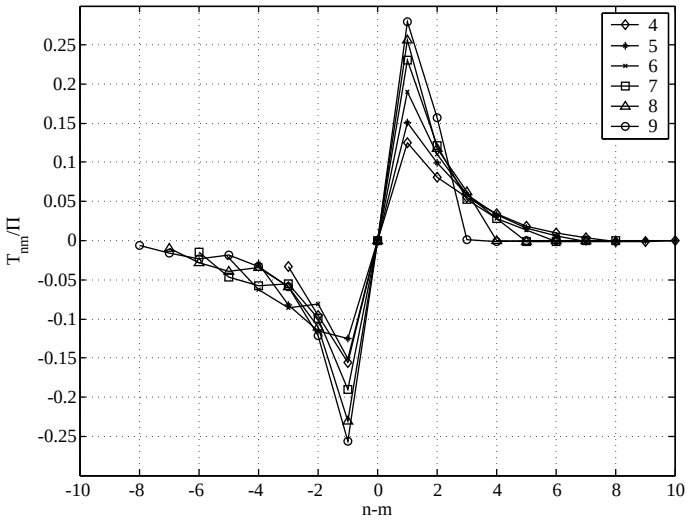


Figure 7: Plot of normalized shell-to-shell energy transfer T_{nm}/Π vs. $n-m$ for $d=3$obtained from numerical simulations on 512^3DNS. The nth shell is $(k_0 s^n : k_0 s^{n+1})$ with $s=2^{1/4}$.The energy transfer is maximum for $n=m\pm 1$, hence the energy transfer is local and self-similar. The energy transfer is also forward. Taken from Verma et al. [56].

where X stands for the field $\mathbf{u}$, $X(\mathbf{p},t)X(\mathbf{q},t)$ represents all the nonlinear terms, and ζ is the dissipation coefficient (ν). The evolution of second and third moment would be

$$\left(\frac{d}{dt}+2\zeta k^2\right)\langle X(\mathbf{k},t)X(-\mathbf{k},t)\rangle = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}}\langle X(-\mathbf{k},t)X(\mathbf{p},t)X(\mathbf{q},t)\rangle \tag{104}$$

$$\left(\frac{d}{dt}+\zeta(k^2+p^2+q^2)\right)\langle X(-\mathbf{k},t)X(\mathbf{p},t)X(\mathbf{q},t)\rangle = \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{s}=\mathbf{0}}\langle X(\mathbf{q},t)X(\mathbf{p},t)X(\mathbf{r},t)X(\mathbf{s},t)\rangle$$

If X were Gaussian, third-order moment would vanish. However, quasi-normal approximation gives nonzero triple correlation; here we replace $\langle XXXX\rangle$ by its Gaussian value, which is a sum of products of second-order moments.

Hence,

$$\langle X(-\mathbf{k},t)X(\mathbf{p},t)X(\mathbf{q},t)\rangle \;=\; \int_0^t d\tau \exp -\zeta(k^2+p^2+q^2)(t-\tau) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} [\langle X(\mathbf{q},\tau)X(-\mathbf{q},\tau)\rangle \langle X(\mathbf{p},\tau)X(\mathbf{p},\tau)\rangle + ...],$$

where ... refers to other products of second-order moments. The substitution of the above in Eq. (104) yields a closed form equation for second-order correlation functions. Orszag [39] discovered that the solution of the above equation was plagued by problems like negative energy. To cure this problem, a suitable linear relaxation operator of the triple correlation (denoted by μ) was introduced (Eddy-damped approximation). In addition, it was assumed that the characteristic evolution time of $\langle XX\rangle \langle XX\rangle$ is larger than $\left(\mu_{kpq}+\nu(k^2+p^2+q^2)\right)^{-1}$(Markovian approximation). As a result the following form of energy evolution equation is obtained

$$\left(\frac{d}{dt}+2\zeta k^2\right)\langle X(\mathbf{k},t)X(-\mathbf{k},t)\rangle = \int d\mathbf{p}\theta_{kpq}(t) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} [\langle X(\mathbf{q},t)X(-\mathbf{q},t\rangle \langle X(\mathbf{p},t)X(-\mathbf{p},t\rangle + ...], \tag{105}$$

where

$$\theta_{kpq}(t) = \left(1-\exp -(\mu_k+\mu_p+\mu_q)t\right) / \left(\mu_k+\mu_p+\mu_q\right)$$

with

$$\mu_k = (\nu+\eta)\, k^2 + C_s \left(\int_0^k dq\, (E^u(q))\, q^2\right)^{1/2}. \tag{106}$$

The first and second terms represent viscous and nonlinear eddy-distortion rates respectively. Note that homogeneity and isotropy are assumed in EDQNM analysis too.

The right-hand side of Eq. (105) is very similar to the perturbative expansion of $S^{uu}(k|p|q)$ (under $t\to\infty$). The term μ_k of Eq. (106) is nothing but the renormalized dissipative parameters. Thus, field-theoretic techniques for turbulence is quite similar to EDQNM calculation. There is a bit of difference however. In field-theory, we typically compute asymptotic energy fluxes in the inertial range. On the contrary, energy is numerically evolved in EDQNM calculations.

IX. INTERMITTENCY IN FLUID TURBULENCE

The famous Kolmogorov's turbulence model assumes a constant energy flux or dissipation rate at all scales, i. e., $\Pi(k) \sim \int dk\nu(k)k^2E(k)$ is independent of k. The renormalized viscosity $\nu(k) \sim k^{-4/3}$ and $E(k) \sim k^{-5/3}$ are consistent with the above assumption. Landau however [28] pointed out that the dissipation rate, which is proportional to the square of vorticity, is singular and quite inhomogeneous. Thus Kolmogorov's theory of turbulence needs modification. The above phenomena in which strong dissipation is localized both in time and space is called intermittency.

A. Quantitative Measures of Intermittency

There are several quantitative measures of intermittency. Consider the increment of the velocity, or some other field, between two points separated by $\mathbf{l}$,

$$\delta\mathbf{u}(\mathbf{x},\mathbf{l}) = \mathbf{u}(\mathbf{x}+\mathbf{l}) - \mathbf{u}(\mathbf{x}). \tag{107}$$

The longitudinal component of $\delta\mathbf{u}(\mathbf{x},\mathbf{l})$ will be given by

$$\delta u_{||}(l) = \delta\mathbf{u}(\mathbf{x},\mathbf{l}).\mathbf{l}/l, \tag{108}$$

and the transverse component is $\delta u_{\perp}(l) = \delta\mathbf{u}(\mathbf{x},\mathbf{l}) - \delta u_{||}(l)\mathbf{l}/l$. Here we have assumed homogeneity and isotropy for turbulence, so that the increment in velocities depend only on l, not on $\mathbf{x}$. Now we define longitudinal and transverse structure functions using

$$S^{(n)}(l) = \left\langle \left[\delta u_{||}(l)\right]^n \right\rangle, \quad U^{(n)}(l) = \langle [\delta u_{\perp}(l)]^n \rangle \tag{109}$$

respectively. The structure function $S^{(n)}(l)$ is expected to have a power law behaviour for l in the inertial range,

$$S^{(n)}(l) = a_n l^{\zeta_n}, \tag{110}$$

where a_n and ζ_n are universal numbers. The exponents ζ_n are called the intermittency exponent.

Moments and probability density function (pdf) are equivalent description of random variables. Note that if $P(\delta u_{||}(l))$ were gaussian, i. e.,

$$P(\delta u_{||}(l)) = \frac{1}{\sigma_r\sqrt{\pi}} \exp -\frac{(\delta u_{||}(l))^2}{\sigma_r^2} \tag{111}$$

then, it is easy to verify that

$$\left\langle (\delta u_{||}(l))^n \right\rangle \propto \sigma_r^n. \tag{112}$$

Kolmogorov's model of turbulence predicts that

$$\sigma_r \sim \epsilon^{1/3} l^{1/3}, \tag{113}$$

For constant ϵ, we obtain

$$S^{(n)}(l) \propto \epsilon^{n/3} l^{n/3}. \tag{114}$$

Systems with gaussian probability distribution or equivalently $S^{(n)}(l) \propto l^{cn}$ (c = constant) are called non-intermittent system. For intermittent systems, the tails of pdf decays slower that gaussian, and could follow a powerlaw.

Structure function can be written in terms of local dissipation rate [38]

$$\delta u_l \sim \epsilon_l^{1/3} l^{1/3}. \tag{115}$$

Kolmogorov [22] introduced the *refined similarity hypothesis* relating structure function to ϵ_l as

$$S_{||}^{(n)}(l) = d_n \left\langle \epsilon_l^{n/3} \right\rangle l^{n/3}. \tag{116}$$

If

$$\langle \epsilon_l^n \rangle \sim l^{\mu_n}, \tag{117}$$

then

$$\zeta_n = \frac{n}{3} + \mu_{n/3}. \tag{118}$$

Many researchers have attempted to model ϵ_l.

In any numerical simulation or experiment, the powerlaw range is quite limited. However, when we plot $S^{(n)}(l)$ vs. $S^{(3)}(l)$, we obtain a much larger scaling range. This phenomena is called Extended self-similarity (ESS). Since, $S^{(3)}(l) \propto l$ [21], ζ_n measured using Eq. (110) or ESS are expected to be the same.

There have been some ingenious attempts to theoretically compute the intermittency exponents (e.g., see series of papers by L'vov and Procaccia [31]). Yet, this problem is unsolved. Here we list some of the phenomenological models. For extensive discussions on intermittency refer to Frisch [17].

B. Results on Intermittency in Fluid Turbulence

In fluid turbulence, the pdf of velocity increment deviates from gaussian [17]. In experiments and simulations one finds that ζ_n vs. n is a nonlinear function of n. Hence, fluid turbulence shows intermittency. Note that $\zeta_2 \approx 0.71$, which yields a correction of approximately 0.04 to Kolmogorov's spectral index of $5/3$. However, the correction for large n is much more. See Frisch [17] for further details.

Remarkably, starting from Navier-Stokes equation, Kolmogorov [21] obtained an exact relation

$$S^{(3)}_{||}(l) = -\frac{4}{5}\epsilon l \tag{119}$$

under $\nu \to 0$ limit (also see [17, 28]). Note that ϵ is the mean dissipation rate. Unfortunately, similar relationship could not be derived for other structure functions. In the following discussion we will discuss some of the prominent intermittency models for fluid turbulence.

1. *Kolmogorov's log-normal model*

Obukhov [38] and Kolmogorov [22] claimed that the dissipation rate in turbulent fluid is log-normal. As a consequence,

$$\zeta_n = \frac{n}{3} - \mu\frac{n(n-3)}{18}, \tag{120}$$

where

$$\langle \epsilon(\mathbf{x})\epsilon(\mathbf{x}+\mathbf{l})\rangle \sim l^{-\mu}. \tag{121}$$

Numerical simulations and experiments give $\mu \approx 0.2$.

The predictions of this model agree well with the experimental results up to $n \approx 10$, but fails for higher values of n. In Fig. 8 we have plotted the above ζ_n along with other model predictions given below.

2. *The $\beta-$model*

Novikov and Stewart [37] and Frisch et al. [16] proposed that smaller scales in turbulent fluid is less space filling. In each step of the cascade an eddy δu_n of scale l_n splits into $2^D\beta$ eddies of scale $l_{n+1} = l_n/2$, where D is the space dimensionality, and β is a fixed parameter with $0 < \beta \leq 1$. In this model

$$\zeta_n = \frac{n}{3} - \frac{\delta}{3}(n-3), \tag{122}$$

where $\beta = 2^{-\delta}$.

Note that ζ_n is linear in n, and it does not match with experimental and numerical data for large n (see Fig. 8).

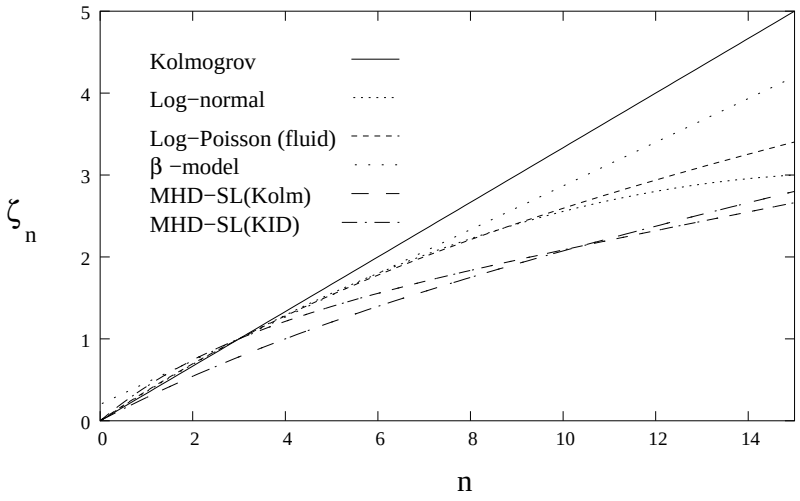


Figure 8: Plots of $\zeta_n$vs. n for various intermittency models in fluids and MHD. She-Leveque's log-Poission model fits best with the experimental data in both fluid and MHD. For MHD turbulence, Kolmogorov-like models are in better agreement than KID's like model.

3. The Multifractal Models

Parisi and Frisch [40] developed a multifractal model of turbulence. Maneveau and Sreenivasan [32] constructed an intuitive model. Here the energy cascade rate ϵ_l is not distributed equally into smaller eddies, say, in each cascade it gets divided into $p\epsilon_l$ and $(1-p)\epsilon_l$. After several cascades, one finds that energy distribution is very skewed or intermittent. The intermittency exponent in this model is

$$\zeta_n = \left(\frac{n}{3} - 1\right) D_n + 1, \tag{123}$$

with

$$D_n = \log_2 \left(p^n + (1-p)^n\right)^{1/(1-n)}. \tag{124}$$

For p near 0.7, ζ_n fits quite well with the experimental data. The deficiency of this model is that it requires an adjustable parameter p. For more detailed discussion, refer to Stolovitzky and Sreenivasan[45].

4. The Log-Poisson Model

She and Leveque [43] proposed a model involving a hierarchy of fluctuating structures associated with the vortex filament. In their model

$$\zeta_n = \frac{n}{3}(1-x) + C_0\left(1 - \beta^{n/3}\right) \tag{125}$$

where C_0 is co-dimension of the dissipative eddies, and x and β are parameters connected by

$$C_0 = \frac{x}{1-\beta} \tag{126}$$

For Kolmogorov scaling, $x = \beta = 2/3$. In hydrodynamic turbulence, the dissipative eddies are vortex filaments, i.e., one-dimensional structures. Therefore, the co-dimension is $C_0 = 2$. Hence, for fluid turbulence

$$\zeta_n^{SL} = \frac{n}{9} + 2\left[1 - \left(\frac{2}{3}\right)^{n/3}\right]. \tag{127}$$

The above prediction fits remarkably well with experimental results. All the above functions have been plotted in Fig. 8 for comparison.

X. CONCLUSION

In this paper we review some of the important results in statistical theory of fluid turbulence. The focus was on field-theoretic techniques applied to fluid turbulence.

Many current topics of interest have not been included in the paper. Some of them are intermittency, large eddy simulations, compressible turbulence etc.

Appendix A: FOURIER SERIES VS. FOURIER TRANSFORM FOR TURBULENT FLOWS

In statistical theory turbulence we typically assume the flow field to be homogeneous. Therefore, Fourier transform is not applicable to these flows in strict sense. However, we can define these quantities by taking limits carefully. This issue has been discussed by Batchelor [1] and McComb [33]. We briefly discuss them here because they form the basis of the whole paper.

A periodic function $\mathbf{u}(\mathbf{x})$ in box L^d can be expanded using Fourier series as following:

$$\mathbf{u}(\mathbf{x}) = \sum \hat{\mathbf{u}}(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x}), \tag{A1}$$

$$\hat{\mathbf{u}}(\mathbf{k}) = \frac{1}{L^d}\int d\mathbf{x}\mathbf{u}(\mathbf{x}) \exp(-i\mathbf{k}\cdot\mathbf{x}), \tag{A2}$$

where d is the space dimensionality. When we take the limit $L \to \infty$, we obtain Fourier transform. Using $\mathbf{u}(\mathbf{k}) = \hat{\mathbf{u}}(\mathbf{k})L^d$, it can be easily shown that

$$\mathbf{u}(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^d}\mathbf{u}(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x}), \tag{A3}$$

$$\mathbf{u}(\mathbf{k}) = \int d\mathbf{x}\mathbf{u}(\mathbf{x}) \exp(-i\mathbf{k}\cdot\mathbf{x}), \tag{A4}$$

with integrals performed over the whole space. Note however that Fourier transform (integral converges) makes sense when $u(x)$ vanishes as $|x| \to \infty$, which is not the case for homogeneous flows. However, correlations defined below are sensible quantities. Using the above equations, we find that

$$\begin{aligned}\langle u_i(\mathbf{k})u_j(\mathbf{k}')\rangle &= \int d\mathbf{x}d\mathbf{x}' \langle u_i(\mathbf{x})u_j(\mathbf{x}')\rangle \exp -i(\mathbf{k}\cdot\mathbf{x} + \dot{\mathbf{k}'}\cdot\mathbf{x}') \\ &= \int d\mathbf{r}C_{ij}(\mathbf{r}) \exp -i\mathbf{k}\cdot\mathbf{r}\int d\mathbf{x}\exp -i(\mathbf{k}+\dot{\mathbf{k}'})\cdot\mathbf{x} \\ &= C_{ij}(\mathbf{k})(2\pi)^d\delta(\mathbf{k}+\mathbf{k}')\end{aligned} \tag{A5}$$

We have used the fact that $\delta(\mathbf{k}) \approx L^d/(2\pi)^d$. The above equation holds the key. In experiments we measure correlation function $C(\mathbf{r})$ which is finite and decays with increasing r, hence spectra $C(\mathbf{k})$ is well defined. Now energy spectrum as well as total energy can be written in terms of $C(\mathbf{k})$ as the following:

$$\begin{aligned}\left\langle u^2\right\rangle = \frac{1}{L^d}\int d\mathbf{x}u^2 = \sum_{\mathbf{k}}|\hat{\mathbf{u}}(\mathbf{k})|^2 &= \frac{1}{L^d}\int\frac{d\mathbf{k}}{(2\pi)^d}\left\langle|\mathbf{u}(\mathbf{k})|^2\right\rangle \\ &= (d-1)\int\frac{d\mathbf{k}}{(2\pi)^d}C(\mathbf{k})\end{aligned} \tag{A6}$$

We have used the fact that $\delta(\mathbf{k}) \approx L^d/(2\pi)^d$. Note that $\left\langle|\mathbf{u}(\mathbf{k})|^2\right\rangle = (d-1)C(\mathbf{k})L^d$ [see Eq. (A5)] is not well defined in the limit $L \to \infty$.

In conclusion, the measurable quantity in homogeneous turbulence is the correlation function, which is finite and decays for large r. Therefore, energy spectra etc. are well defined objects in terms of Fourier transforms of correlation functions.

We choose a finite box, typically $(2\pi)^d$, in spectral simulations for fluid flows. For these problems we express the equations *(incompressible* MHD) in terms of Fourier series. We write them below for reference.

$$\left(\frac{\partial}{\partial t} + \nu k^2\right)\hat{u}_i(\mathbf{k},t) = -ik_i\hat{p}_{tot}(\mathbf{k},t) - ik_j\sum[\hat{u}_j(\mathbf{q},t)\hat{u}_i(\mathbf{p},t)] \tag{A7}$$

The energy spectrum can be computed using $\hat{u}_i(\mathbf{k},t)$:

$$\int E(k)dk = \sum|\hat{\mathbf{u}}(\mathbf{k})|^2/2 = \int d\mathbf{n}|\hat{\mathbf{u}}(\mathbf{k})|^2/2 = \int d\mathbf{k}|\hat{\mathbf{u}}(\mathbf{k})|^2/2 \tag{A8}$$

where $\mathbf{n}$ is the lattice vector in d-dimensional space. The above equation implies that

$$E(k) = \frac{|\hat{\mathbf{u}}(\mathbf{k})|^2}{2} S_d k^{d-1}. \tag{A9}$$

A natural question is why the results of numerical simulations or experiments done in a finite volume should match with those obtained for infinite volume. The answer is straight forward. When we go from size 2π to L, the wavenumbers should be scaled by $(2\pi)/L$. The velocity and frequency should be should be scaled by $(2\pi)/L$ and $[(2\pi)/L]^2$ to keep dimensionless ν fixed. The evolution of the two systems will be identical apart from the above factors. Hence, numerical simulations in a box of size 2π can capture the behaviour of a system with $L \to \infty$, for which Fourier transform in defined.

Appendix B: PERTURBATIVE CALCULATION OF NAVIER STOKES EQUATION

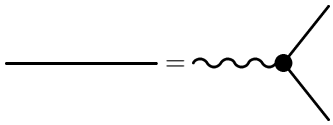
The Navier Stokes can be written as

$$u_i(\hat{k}) = G(\hat{k}) - \frac{i}{2} P^+_{ijm}(\mathbf{k}) \int d\hat{p}[u_j(\hat{p}) u_m(\hat{k}-\hat{p})] \tag{B1}$$

where the Greens function G can be written as

$$G^{-1}(k,\omega) = -i\omega - \Sigma^{uu} \tag{B2}$$

We solve the above equation perturbatively keeping the terms upto the first nonvanishing order. The corresponding Feynmann diagram is



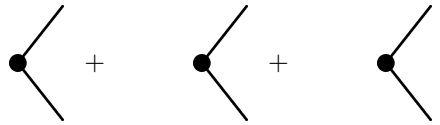
(B3)

The solid line represents fields u, and the wiggly line (photon) denotes G. The filled circle denotes $-(i/2)P^+_{ijm}$ vertex. These diagrams appear in renormalization calculations as well as in energy flux calculations.

1. Viscosity Renormalization

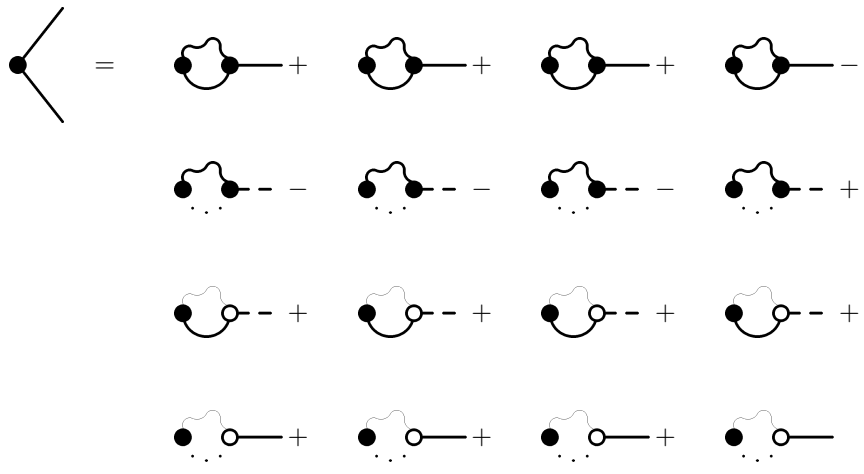
The expansion of u in terms of Feynman diagrams are given below:

I^u =



(B4)

Factor of 2 appears in I^u because of $<>$ symmetry in the corresponding term. To zeroth order, the terms with $<>$ are zero because of quasi-gaussian nature of $>$ modes. To the next order in perturbation, the third term of I^u is



+higer oder diagrams (B5)

In the above diagrams solid lines denote $< u(\mathbf{k})u(\mathbf{k}') >$. As mentioned earlier, the wiggly line denotes Green's functions. All the diagrams except 4,8,12, and 16th can be shown to be trivially zero using Eqs. (73,74). We assume that 4,8,12, and 16th diagrams are also zero, as usually done in turbulence RG calculations [33, 58, 64, 65]. Hence, the term is zero. Now we are left with >> terms (3rd term of I^u), which is

$$I_3^u \;=\; = -\delta\Sigma(k) \tag{B6}$$

where

$$-(d-1)\delta\Sigma \;=\; \tag{B7}$$

In the above equation we have omitted all the vanishing diagrams (similar to those appearing in Eq. [B5]). These terms contribute to Σs.

The algebraic expressions for the above diagrams are given in Section VII. For isotropic flows, the algebraic factors $S(k,p,q)$ resulting from tensor contractions are given below.

$$S(k,p,q) \;=\; P^+_{bjm}(k)P^+_{mab}(p)P_{ja}(q) = kp\left((d-3)z + 2z^3 + (d-1)xy\right) \tag{B8}$$

In the next subsection we will derive the terms for mode-to-mode energy transfer function.

2. Mode-to-Mode Energy Transfer in fluid Turbulence

In Section 3, we studied the "mode-to-mode" energy transfer $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ from mode **p** to mode **k'**, with mode **q** acting as a mediator. The perturbative calculation of S involves the following terms

$$\langle S^{uu}(k'|p|q)\rangle \;=\; \tag{B9}$$

In all the diagrams, the left vertex denotes k_i, while the filled circle of the right vertex represent $(-i/2)P^+_{ijm}$. For

isotropic nonhelical flows, the algebraic factors are given below. The factors for the diagrams are T_1, T_2, T_3 in sequential order.

$$T_1(k,p,q) = k_i P^+_{jab}(k) P_{ja}(p) P_{ib}(q) = kp\left((d-3)z + (d-2)xy + 2z^3 + 2xyz^2 + x^2 z\right) \quad \text{(B10)}$$

$$T_3(k,p,q) = -k_i P^+_{jab}(p) P_{ja}(k) P_{ib}(q) = -kp\left((d-3)z + (d-2)xy + 2z^3 + 2xyz^2 + y^2 z\right) \quad \text{(B11)}$$

$$T_3(k,p,q) = -k_i P^+_{iab}(q) P_{ja}(k) P_{jb}(p) = -kq\left(xz - 2xy^2 z - yz^2\right) \quad \text{(B12)}$$

$$\text{(B13)}$$

These terms are similar to those given in Leslie [30].

Appendix C: MODE-TO-MODE ENERGY TRANSFER IN SCALAR TURBULENCE

The equations for passive scalar turbulence are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u}, \quad \text{(C1)}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = \kappa\nabla^2\theta, \quad \text{(C2)}$$

$$\nabla\cdot\mathbf{u} = 0, \quad \text{(C3)}$$

where θ is the scalar density. Note that the scalar is convected by velocity field, but the scalar does not affect the flow. For details refer to Lesieur [30] and Stanišić [46].

For energy transfer in scalar field, we can follow the same procedure as in fluid turbulence. If we take only a single triad $(\mathbf{k}', \mathbf{p}, \mathbf{q})$ with $\mathbf{k}' + \mathbf{p} + \mathbf{q} = \mathbf{0}$, energy is conserved for $\kappa = 0$. Following fluid turbulence, we can show that the energy equation for scalar turbulence can be written in terms of 'mode-to-mode energy transfer' $S^{\theta\theta}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ from mode $\theta(\mathbf{p})$ to $\theta(\mathbf{k}')$ with $\theta(\mathbf{q}), \mathbf{u}(\mathbf{q})$ as a mediator, which is

$$S^{\theta\theta}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = -\Im\left([\mathbf{k}'.\mathbf{u}(\mathbf{q})]\,[\theta(k')\theta(p)]\right) \quad \text{(C4)}$$

where $\Im$ stands for the imaginary part of the argument. The energy equation for scalar field is

$$\left(\frac{\partial}{\partial t} + 2\kappa k^2\right) C^\theta(\mathbf{k}) = [S^{\theta\theta}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + S^{\theta\theta}(\mathbf{k}'|\mathbf{q}|\mathbf{p})] \quad \text{(C5)}$$

Note that there is no cross-transfer between u and ψ energy. It is also important to note that both C^u and C^ψ are conserved in every triad interaction, i.e.,

$$S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + S^{uu}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) + S^{uu}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) + S^{uu}(\mathbf{p}|\mathbf{q}|\mathbf{k}') + S^{uu}(\mathbf{q}|\mathbf{k}'|\mathbf{p}) + S^{uu}(\mathbf{q}|\mathbf{p}|\mathbf{k}') = 0 \quad \text{(C6)}$$

$$S^{\psi\psi}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + S^{\psi\psi}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) + S^{\psi\psi}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) + S^{\psi\psi}(\mathbf{p}|\mathbf{q}|\mathbf{k}') + S^{\psi\psi}(\mathbf{q}|\mathbf{k}'|\mathbf{p}) + S^{\psi\psi}(\mathbf{q}|\mathbf{p}|\mathbf{k}') = 0 \quad \text{(C7)}$$

These are the statements of "detailed conservation of energy" in triad interaction (when $\nu = \kappa = 0$) [30].

The energy flux Π^ψ from a wavenumber sphere of radius k_0 is [7]

$$\Pi^\psi(k_0) = \int_{k'>k_0} \frac{d\mathbf{k}'}{(2\pi)^d} \int_{p<k_0} \frac{d\mathbf{p}}{(2\pi)^d} \left\langle S^{\psi\psi}(\mathbf{k}'|\mathbf{p}|\mathbf{q})\right\rangle \quad \text{(C8)}$$

Appendix D: MODE-TO-MODE ENERGY TRANSFERS IN RAYLEIGH BÉNARD CONVECTION

One of the most studied model of convection is Rayleigh Bénard (RB) Convection. In this model the fluid confined between two parellel plates is heated from below. Constant temperature is maintained across these plates. Nondimensionalized equations for RB model are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\sigma + RP\theta\hat{\mathbf{z}} + P\nabla^2\mathbf{u}, \quad \text{(D1)}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = u_3 + \nabla^2\theta, \quad \text{(D2)}$$

where the nondimensinal parameters R and $P = \nu/\kappa$ are Rayleigh and Prandtl numbers respectively. The fluid is assumed to incompressible except for the buoyancy term.

We can apply 'mode-to-mode energy transfer' model to RB convection. The above hydrodynamical equations in Fourier space are

$$\frac{\partial u_i(\mathbf{k})}{\partial t} = -ik_i\sigma\mathbf{k} - ik_j \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{q})u_i(\mathbf{p}) + RP\theta(\mathbf{k})\delta_{i3} - Pk^2u_i(\mathbf{k}) \tag{D3}$$

$$\frac{\partial \theta(\mathbf{k})}{\partial t} = u_3(k) - ik_j \sum_{\mathbf{p}+q=k} u_j(\mathbf{q})\theta(\mathbf{p}) - k^2\theta(\mathbf{k}), \tag{D4}$$

$$k_iu_i(\mathbf{k}) = 0, \tag{D5}$$

where $\mathbf{k} = \mathbf{p} + \mathbf{q}$. We can derive interesting results by focussing on a single triad $(\mathbf{k}', \mathbf{p}, \mathbf{q})$ such that $\mathbf{k}' + \mathbf{p} + \mathbf{q} = 0$. Clearly $\mathbf{k}' = -\mathbf{k}$. We can easily derive the following energy equations:

$$\frac{\partial}{\partial t}\frac{|\mathbf{u}(\mathbf{k})|^2}{2} = S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + S^{uu}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) + RP\Re\left[\theta(\mathbf{k})u_3(\mathbf{k}')\right] - 2Pk^2\frac{|\mathbf{u}(\mathbf{k})|^2}{2}, \tag{D6}$$

$$\frac{\partial}{\partial t}\frac{|\theta(\mathbf{k})|^2}{2} = S^{\theta\theta}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + S^{\theta\theta}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) + \Re\left[\theta(\mathbf{k})u_3(\mathbf{k}')\right] - 2k^2\frac{|\theta(\mathbf{k})|^2}{2}, \tag{D7}$$

where

$$S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = -\Im\left([\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})]\,[\mathbf{u}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{q})]\right), \tag{D8}$$

$$S^{\theta\theta}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = -\Im\left([\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})]\,[\theta(\mathbf{k}')\cdot\theta(\mathbf{q})]\right). \tag{D9}$$

Here $\Re$ and $\Im$ represents the real and imaginary part of the argument. The quantity $S^{\theta\theta}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ represents the energy transfer from mode $\theta(\mathbf{p})$ (the field variable with the second argument) to mode $\theta(\mathbf{k}')$ (the field variable with the first argument) with the help of the mode $\theta(\mathbf{q})$ (the field variable with the third argument) acting as a mediator. The above formalism is called "*mode-to-mode*" formalism for energy transfer [7, 53], and it differs from Kraichnan's "*combined energy transfer*" formalism [23, 29]. The energy equation can be interpreted as follows: The field variables with wavenumber $\mathbf{k}'$ $[\mathbf{u}(\mathbf{k}'),\ \theta(\mathbf{k}')]$ receives energy from the modes $\mathbf{p}$ and $\mathbf{q}$ through mode-to-mode energy transfer terms, and it also receives energy due to interaction term $\theta(\mathbf{k})u_3(\mathbf{k}')$.

It is important to keep in mind that $|\theta(\mathbf{k})|^2/2$ is not real energy. However, it has structure of energy, and we can apply the energy transfer formalism here as well. This idea has been exploited heavily in the past to infer the direction of flux of passive scalar etc. [24]. Dar et al. [7] and Verma [53] have shown that the sum of all energy transfer rates along u-u and $\theta - \theta$ channels are zero, i.e.,

$$S^{XX}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + S^{XX}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) + S^{XX}(\mathbf{p}|k'|q) \tag{D10}$$

$$+S^{XX}(\mathbf{p}|q|k) + S^{XX}(\mathbf{q}|k'|p) + S^{XX}(\mathbf{q}|p|k') = 0, \tag{D11}$$

where XX could be uu or $\theta\theta$.

Using this identity we can easily show that without viscous and thermal diffusion

$$\frac{\partial}{\partial t}|\mathbf{u}(\mathbf{k})|^2 + |\mathbf{u}(\mathbf{p})|^2 + |\mathbf{u}(\mathbf{q})|^2 = 2RP\Re[\theta(\mathbf{k})u_3^*(\mathbf{k}) + \theta(\mathbf{p})u_3^*(\mathbf{p}) + \theta(\mathbf{q})u_3^*(\mathbf{q})], \tag{D12}$$

$$\frac{\partial}{\partial t}\left[|\theta(\mathbf{k})|^2 + |\theta(\mathbf{p})|^2 + |\theta(\mathbf{q})|^2\right] = 2\Re[\theta(\mathbf{k})u_3^*(\mathbf{k}) + \theta(\mathbf{p})u_3^*(\mathbf{p}) + \theta(\mathbf{q})u_3^*(\mathbf{q})], \tag{D13}$$

The interpretation of the above equations is that the triads $[\mathbf{u}(\mathbf{k}'), \mathbf{u}(\mathbf{p}), \mathbf{u}(\mathbf{q})]$ and $[\theta(\mathbf{k}'), \theta(\mathbf{p}), \theta(\mathbf{q})]$ exchange energy between each other via $\theta(\mathbf{k})u_3^*(\mathbf{k})$ interaction terms. The mode-to-mode interaction conserves energy within a triad. The viscous and diffusive terms dissipate kinetic energy and θ-energy respectively.

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[1] G. K. Batchelor. *The Theory of Homogeneous Turbulence*. Cambridge University Press, Cambridge, 1971.

[2] J. K. Bhattacharjee. Homogeneous isotropic turbulence in randomly stirred fluids: Self-consistent mode coupling and perturbation theory. *Phys. Fluids A*, 3:879, 1991.

[3] D. Biskamp. *Magnetohydrodynamic Turbulence*. Cambridge University Press, Cambridge, 2003.

[4] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zhang. *Spectral Methods in Fluid Turbulence*. Springer-Verlag, Berlin, 1988.

[5] D. Carati. Renormalization-group theory of turbulence: a d-dimensional expansion. *Phys. Rev. A*, 41:3129–3133, 1990.

[6] G. Dar, M. K. Verma, and V. Eswaran. A new approach to study energy transfer in fluid and magnetohydrodynamic turbulence. *physics/0006012*, 2000.

[7] G. Dar, M. K. Verma, and V. Eswaran. Energy transfer in two-dimensional magnetohydrodynamic turbulence. *Physica D*, 157:207, 2001.

[8] P. A. Davidson. *Turbulence*. Oxford University Press, Oxford, 2004.

[9] C. DeDominicis and P. C. Martin. Energy spectra of certain randomly-stirred fluids. *Phys. Rev. A*, 19:419–422, 1979.

[10] J. A. Domaradzki and R. S. Rogallo. Local energy transfer and nonlocal interactions in homogenous, isotropic turbulence. *Phys. Fluids A*, 2:413, 1990.

[11] G. L. Eyink. The renormalization group method in statistical hydrodynamics. *Phys. Fluids*, 6:3063, 1994.

[12] D. Forster, D. R. Nelson, and M. J. stephen. Large-distance and long-time properties of a randomly stirred fluids. *Phys. Rev. A*, 16:732–749, 1977.

[13] J. D. Fournier and U. Frisch. d-dimensional turbulence. *Phys. Rev. A*, 17:747–762, 1979.

[14] J. D. Fournier and U. Frisch. Remarks on the renormalization group in statistical fluid dynamics. *Phys. Rev. A*, 28:1000, 1983.

[15] J. D. Fournier, U. Frisch, and H. A. Rose. Infinite-dimensional turbulence. *J. Phys. A*, 11:187, 1978.

[16] U. Frisch. A simple dynamical model of intermittent fully developed turbulence. *J. Fluid Mech.*, 87:719, 1978.

[17] U. Frisch. *Turbulence*. Cambridge University Press, Cambridge, 1995.

[18] T. Gotoh. Small-scale statistics of turbulence at high reynolds numbers by massive compuation. *Comput. Phys. Commun.*, 147:530–532, 2002.

[19] A. N. Kolmogorov. Dissipation of energy in locally in locally isotropic turbulence. *Dokl. Akad. Nauk SSSR*, 32:16–18, 1941.

[20] A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large reynolds number. *Dokl. Akad. Nauk SSSR*, 30:9–13, 1941.

[21] A. N. Kolmogorov. On degeneration (decay) of isotropic turbulence in an incompressible viscous liquid. *Dokl. Akad. Nauk SSSR*, 31:538–540, 1941.

[22] A. N. Kolmogorov. A refinement of previous hypothesis concerning the local structure of turbulence in a viscous incompressible fluid in high reynolds number. *J. Fluid Mech.*, 13:82, 1962.

[23] R. H. Kraichnan. The structure of isotropic turbulence at very high reynolds numbers. *J. Fluid Mech.*, 5:497, 1959.

[24] R. H. Kraichnan. Inertial-range transfer in two- and three-dimensional turbulence. *J. Fluid Mech.*, 47:525, 1971.

[25] R. H. Kraichnan. An interpretation of the yakhot-orszag turbulence theory. *Phys. Fluids*, 30:2400, 1987.

[26] R. H. Kraichnan and S. Chen. Is there a statistical mechanics of turbulence? *Physica D*, 37:160, 1989.

[27] R. H. Kraichnan and D. Montgomery. Two-dimensional turbulence. *Phys. Rep.*, 43:547–619, 1980.

[28] L. D. Landau and E. M. Lifsitz. *Fluid Mechanics*. Pergamon Press, Oxford, 1987.

[29] M. Lesieur. *Turbulence in Fluids - Stochastic and Numerical Modelling*. Kluwer Academic Publishers, Dordrecht, 1990.

[30] D. C. Leslie. *Development in the Theory of Turbulence*. Oxford University Press, Claredon, 1973.

[31] V. S. L'vov and I. Procaccia. Intermittency in hydrodynamic turbulence as intermediate asymptotics to kolmogorov scaling. *Phys. Rev. Lett.*, 74:2690, 1995.

[32] C. Maneveau and K. R. Sreenivasan. Simple multifractal cascade model for fully developed turbulence. *Phys. Rev. Lett.*, 59:1424–1427, 1987.

[33] W. D. McComb. *The Physics of Fluid Turbulence*. Oxford University Press, Claredon, 1990.

[34] W. D. McComb. Theory of turbulence. *Rep. Prog. Phys.*, 58:1117–1206, 1995.

[35] W. D. McComb. *Renormalization Methods*. Clarendon Press, Oxford, 2004.

[36] W. D. McComb and A. G. Watt. Two-field theory of incompressible-fluid turbulence. *Phys. Rev. A*, 46:4797, 1992.

[37] E. A. Novikov and R. W. Stewart. The intermittency of turbulence and the spectrum of energy dissipation. *Izv. Akad. Nauk SSSR, Ser. Geoffiz*, 3:408–412, 1964.

[38] A. M. Obukhov. Some specific features of atmospheric turbulence. *J. Fluid Mech.*, 13:77, 1962.

[39] S. A. Orszag. Statistical theory of turbulence. In *Fluid Dynamics, Les Houches 1973 Summer School of Theoretical Physics*, page 273. Gordon and Breach, Berlin, 1977.

[40] G. Parisi and U. Frisch. On the singularity structure of fully developed turbulence. In M. Ghil, R. Benzi, and G. Parisi, editors, *Turbulence and Predictibility in Geophysical Fluid Dynamics*, pages 744–745, Amsterdam, 1985. North-Holland.

[41] S. B. Pope. *Turbulent Flows*. Cambridge University Press, Cambridge, 2000.

[42] A. Pouquet. On two-dimensional magnetohydrodynamic turbulence. *J. Fluid Mech.*, 88:1–16, 1978.

[43] Z. S. She and E. Leveque. Universal scaling laws in fully developed turbulence. *Phys. Rev. Lett.*, 72:336, 1994.

[44] L. M. Smith and S. L. Woodruff. Renormalization-group analysis of turbulence. *Ann. Rev. Fluid Mech.*, 30:275–310, 1998.

[45] K. R. Sreenivasan. Simple multifractal cascade model for fully developed turbulence. *Rev. Mod. Phys.*, 71:S383–S395, 1999.

[46] M. M. Stanišić. *Mathematic Theory of Turbulence*. Springer-Verlag, New York, 1988.

[47] E. V. Teodorovich. On the calculation of the kolmogorov constant in a description of turbulence by means of the renormalization method. *Sov. Phys. JETP*, 69:89, 1989.

[48] D. J. Tritton. *Physical Fluid Dynamics*. Clarendon Press, Oxford, 1988.

[49] M. K. Verma. Mean magnetic field renormalization and kolmogorov's energy spectrum in magnetohydrodynamic turbulence. *Phys. Plasmas*, 6:1455, 1999.

[50] M. K. Verma. Computation of renormalized viscosity and resistivity in magnetohydrodynamic turbulence. *Phys. Plasmas*, 8:3945–3956, 2001.
[51] M. K. Verma. Field theoretic calculation of renormalized-viscosity, renormalized-resistivity, and energy fluxes of magnetohydrodynamic turbulence. *Phys. Rev. E*, 64:26305, 2001.
[52] M. K. Verma. Field theoretic calculation of scalar turbulence. *Int. J. Mod. Phys. B*, 15:3419, 2001.
[53] M. K. Verma. Statistical theory of magnetohydrodynamic turbulence: Recent results. *Phys. Rep.*, 401:229–380, 2004.
[54] M. K. Verma. Incompressible turbulence as nonlocal field theory. *Pramana*, 64:333, 2005.
[55] M. K. Verma, A. Ayyer, and A. V. Chandra. Energy transfers and locality in magnetohydrodynamic turbulence. *Phys. Plasmas*, 12:82307, 2005.
[56] M. K. Verma, A. Ayyer, O. Debliquy, S. Kumar, and A. V. Chandra. Local shell-to-shell energy transfer via nonlocal interactions in fluid turbulence. *Pramana, J. Phys.*, 65:297, 2005.
[57] K. Wilson and J. Kogut. The renormalization group and ϵ expansion. *Phys. Rep.*, 12:75, 1974.
[58] V. Yakhot and S. A. Orszag. Renormalization group analysis of turbulence. i. basic theory. *J. Sci. Comput.*, 1:3, 1986.
[59] G. P. Zank and W. H. Matthaeus. Nearly incompressible hydrodynamics and heat conduction. *Phys. Rev. Lett.*, 64:1243–1246, 1990.
[60] G. P. Zank and W. H. Matthaeus. The equation of nearly incompressible fluids. i. hydrodynamics, turbulence, and waves. *Phys. Fluids A*, 3:69–82, 1991.
[61] Y. Zhou. Effect of helicity on renormalized eddy viscosity and subgrid scale closure for hydrodynamic turbulence. *Phys. Rev. A*, 41:5683–5686, 1990.
[62] Y. Zhou. Degree of locality of energy transfer in the inertial range. *Phys. Fluids*, 5:1092–1094, 1993.
[63] Y. Zhou and W. D. McComb. Renormalization group (rg) in turbulence: Historical and comparative perspective. *ICASE Rep. No. 97-36*, (NASA CR-201718), 1997.
[64] Y. Zhou and G. Vahala. Reformulation of recursive-renormalization-group-based subgrid modeling of turbulence. *Phys. Rev. E*, 47:2503, 1993.
[65] Y. Zhou, G. Vahala, and M. Hussain. Renormalization-group theory for the eddy viscosity in subgrid modeling. *Phys. Rev. A*, 37:2590, 1988.