Large eddy simulations using recursive renormalization-group based eddy viscosity

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We have used renormalized viscosity derived using 'self-consistent' recursive renormalization-group method to perform large eddy simulations (LES) of decaying homogeneous and isotropic turbulence inside a periodic cubical box on coarse grids (32^3, 64^3 and 128^3) at initial Taylor Reynolds number, $\mathrm{R}_\lambda = 315$. The results from LES were compared against direct numerical simulation (DNS) results (512^3 grid) at the same initial R_λ. There is a good agreement between the computed quantities for LES and DNS - temporal evolution of turbulence kinetic energy $\mathrm{E_t}$, kinetic energy spectra $\mathrm{E_u(k)}$, kinetic energy flux $\Pi_\mathrm{u}(\mathrm{k})$- and the evolution of large scale structures, visualized using the velocity magnitude and finite-time-Lyapunov-exponent isosurfaces, too remain similar for both classes of the simulations. This establishes the suitability of using recursive renormalization-group based eddy viscosity in performing large eddy simulations.

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I. INTRODUCTION

Owing to the scaling $N \propto Re^{9/4}$, where N is the total number of grid points in a 3-D simulation and Re is the Reynolds Number, resolution of all the scales in numerical simulations of turbulent flows, so called direct numerical simulations (DNS), is limited to moderate Reynolds numbers. Large eddy simulation (LES) is the most efficient technique for simulating high Re flows. In LES, one does not resolves the small dissipating scales (sub-grid scales) explicitly and hence these must be modeled. There are several sub-grid scale (SGS) models[1]. The earliest SGS model used was the Smagorinsky model[2] where the effect of small scales is modeled by using an eddy viscosity.

Certain issues with Smagorinsky model such as unconditional dissipation, neglect of backscatter and empirical nature of the constants involved are addressed in the dynamic version of the Smagorinsky model[3] where an algebraic identity between the sub-grid scale stresses at two different filtered levels and the resolved turbulent stresses is utilized to evaluate the model constant and some backscatter is also captured. Dynamic models, however, introduce numerical instability without proper averaging of the terms involved. This averaging itself becomes challenging for flows around complex geometries. Another class of models are scale-similar[4] models which presume the similarity between the structure of velocity field above and below the cutoff used to distinguish sub-grid scales from super-grid scales. Several other models have emerged, where instead of modeling the SGS tensor, the SGS velocity field is modeled[5–7]. In the work of Misra and Pulin[7], for example, the SGS structure of the turbulence is assumed to consist of stretched vortices whose orientations are determined by the resolved velocity field. The implied velocity field is used to evaluate the SGS stress tensor. In spite of many such SGS models, no model can boast of accurately capturing the effect of sub-grid scales for a vast range of flow situations. To remedy this, models which can better integrate turbulence theory and simulations are constantly sought.

Renormalization-group (RNG), a popular-technique among physicists, has gained attention in tackling the physics of turbulent flows and looks promising in the development of SGS models which are better integrated with theoretical arguments in turbulence. Broadly, one can divide RNG approaches into two categories: ϵ-expansion method and the recursive method. In the former approach[8,9] a zero mean Gaussian random forcing, of the form $\hat{F}(k) = D_0 k^{-d+(4-\epsilon)}$, is introduced into the Navier-Stokes equations. Here ϵ is a small parameter introduced in the exponent of the power law. Upon removing the sub-grid wavenumber shells, higher-order non-linear terms are introduced. For $\epsilon \ll 1$, these terms turn out to be "irrelevant". Yakhot and Orszag[10] evaluated important constants of turbulent flows such as the Kolmogorov constant, turbulent Prandtl number, Batchelor constant using ϵ-expansion approach. Yakhot and Orszag have [11] further used the evaluated eddy viscosity for LES of wall bounded flows. However, in all this, they used $\epsilon = 4$ and still do away with the high order non-linear terms. This is quiet ambiguous and mathematically inconsistent. Eyink[12] has further shown that such high-order nonlinear terms are not "irrelevant" but marginal by power counting and can not be neglected even for small ϵ.

In the recursive approach[13], however, one successively eliminates the sub-grid wavenumber shells to obtain an integro-difference recursive relation for the eddy viscosity in the super-grid range. Upon application of RNG to this recursive relation, the eddy viscosity tends to a wavenumber dependent fixed point. In doing so, unlike the ϵ-expansion RNG scheme, no expansion parameter ϵ is needed and hence the ambiguity involved in eliminating the high order non-linear terms while keeping a high value of the perturbation parameter is not tackled. Thus, the recursive RNG approach sounds mathematically more consistent in tackling turbulent flows. Motivated by the success of Yakhot and Orszag's[11] simulations using ϵ-expansion based renormalized viscosity and the mathematical consistency of recursive RNG approach, one is tempted to use renormalized viscosity, calculated using recursive method, for LES. McComb and Watt[14] too had suggested such an approach. However, apart from a preliminary investigation of Shishir and Verma[15], no detailed investigation into this issue has been performed. In the present work, we have performed such an investigation and have used renormalized viscosity derived using the recursive RNG method to perform LES of decaying, homogeneous and isotropic turbulence inside a periodic cubical box.

The outline of the paper is as follows: In Sec. II we discuss the details regarding the governing equations and the renormalized viscosity used in LES. Computational methodologies are discussed in Sec. III. Results obtained from LES and DNS are discussed in Sec. IV. Finally, we summarize our results in Sec. V.

II. MATHEMATICAL FORMULATIONS

For turbulence in a cubical box with side L, the real-space velocity $\mathbf{u}(\mathbf{x}, \mathrm{t})$ and pressure $\mathbf{p}(\mathbf{x}, \mathrm{t})$ can be written in Fourier-space as,

$$\mathbf{u}(\mathbf{x}, \mathrm{t}) = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k}, \mathrm{t}) \mathrm{e}^{\mathbf{ik.x}} \tag{1}$$

$$\mathbf{p}(\mathbf{x}, \mathrm{t}) = \sum_{\mathbf{k}} \hat{\mathbf{p}}(\mathbf{k}, \mathrm{t}) \mathrm{e}^{\mathbf{ik.x}} \tag{2}$$

Here, wavenumber vectors $\mathbf{k}$ are integral multiples of $\mathrm{k}_0 = 2\pi/\mathrm{L}$. Using the above formulations, the Navier-Stokes equations, for incompressible flows, in Fourier space can be written as[16],

$$(\frac{\mathrm{d}}{\mathrm{dt}} + \nu_0 \mathrm{k}^2)\hat{\mathrm{u}}_\mathrm{j}(\mathbf{k}, \mathrm{t}) = -\mathrm{ik}_\mathrm{l} \mathrm{P}_{\mathrm{jk}}(\mathbf{k}) \sum_{\mathbf{k}', \mathbf{k}''} \delta_{\mathbf{k}, \mathbf{k}' + \mathbf{k}''} \hat{\mathrm{u}}_\mathrm{j}(\mathbf{k}')\hat{\mathrm{u}}_\mathrm{k}(\mathbf{k}'') \tag{3}$$

where $\mathrm{P}_{\mathrm{jk}}(\mathbf{k}) = -(\delta_{\mathrm{jk}} - \frac{\mathrm{k_j k_k}}{\mathrm{k}^2})$ is the projection tensor and right hand side of Eq. 3 represents the triadic interactions among the wavenumbers $\mathbf{k}', \mathbf{k}'', \mathbf{k}$ such that $\mathbf{k}' + \mathbf{k}'' = \mathbf{k}$. Kroneckar Delta $\delta_{\mathbf{k}, \mathbf{k}' + \mathbf{k}''}$ takes care that only such traidic interactions are captured.
For a sharp spectral filter with cutoff wavenumber $\mathrm{k_c}$, the Fourier coefficients of the filtered velocity field are,

$$\hat{\bar{\mathbf{u}}}(\mathbf{k}, \mathrm{t}) = \mathrm{H}(\mathrm{k_c} - \mathrm{k})\hat{\mathbf{u}}(\mathbf{k}, \mathrm{t}), \tag{4}$$

where H represents Heaviside function, $\mathrm{k} = |\mathbf{k}|$ is the magnitude of wavenumber. The Fourier series for the filtered velocity is then,

$$\bar{\mathbf{u}}(\mathbf{x}, \mathrm{t}) = \sum_{\mathbf{k}} \mathrm{e}^{\mathbf{ik.x}} \hat{\bar{\mathbf{u}}}(\mathbf{k}, \mathrm{t}) = \sum_{|\mathbf{k}| < |\mathbf{k_c}|} \mathrm{e}^{\mathbf{ik.x}} \hat{\mathbf{u}}(\mathbf{k}, \mathrm{t}) \tag{5}$$

Once we substitute for the filtered velocities, we obtain a finite set of ordinary differential equations ($\mathrm{k} < \mathrm{k_c}$), from the preceding infinite set of Eq. 3, given as,

$$(\frac{\mathrm{d}}{\mathrm{dt}} + \nu_0 \mathrm{k}^2)\hat{\bar{\mathrm{u}}}_\mathrm{j}(\mathbf{k}, \mathrm{t}) = -\mathrm{ik}_\mathrm{l} \mathrm{P}_{\mathrm{jk}}(\mathbf{k}) \sum_{\mathbf{k}', \mathbf{k}''} \delta_{\mathbf{k}, \mathbf{k}' + \mathbf{k}''} \mathrm{H}(\mathrm{k_c} - \mathrm{k}) \hat{\mathrm{u}}_\mathrm{k}(\mathbf{k}', \mathrm{t})\hat{\mathrm{u}}_\mathrm{l}(\mathbf{k}'', \mathrm{t}) \tag{6}$$

Note here that this is not a closed system of equations since it includes unknown Fourier coefficients $\hat{\mathrm{u}}_\mathrm{k}(\mathbf{k}', \mathrm{t})$ and $\hat{\mathrm{u}}_\mathrm{l}(\mathbf{k}'', \mathrm{t})$ for $|\mathbf{k}'| > \mathrm{k_c}$ or $|\mathbf{k}''| > \mathrm{k_c}$. LES in wavenumber space aims at modeling these terms. For the large eddy simulations presented in this paper, the effect of such truncated modes on the lower wavenumber modes is accounted for by using renormalized viscosity[17], derived using recursive renormalization-group formalism and is given as,

$$\nu_{\mathrm{r}}(\mathrm{k_c}) = (\mathrm{K_0})^{1/2}\Pi^{1/3}\mathrm{k_c}^{-4/3}\nu^* \tag{7}$$

Here ν_r stands for renormalized viscosity, $\mathrm{K_0}$ is the Kolmogorov constant, Π is the kinetic energy flux in the inertial range of wavenumbers and ν^*, as given in Shishir and Verma[15], is assigned a value of 0.38. Note that renormalized viscosity is a function of cutoff wavenumber. Thus the equations to be solved for LES are given as,

$$(\frac{\mathrm{d}}{\mathrm{dt}} + \nu_{\mathrm{e}}\mathrm{k}^2)\hat{\mathrm{u}}_{\mathrm{j}}(\mathbf{k},\mathrm{t}) = -\mathrm{i}\mathrm{k_l}\mathrm{P_{jk}}(\mathbf{k}) \sum_{|\mathbf{k}'|,|\mathbf{k}''|<|\mathbf{k_c}|} \delta_{\mathbf{k},\mathbf{k}'+\mathbf{k}''}\mathrm{H}(\mathrm{k_c}-\mathrm{k})\hat{\mathrm{u}}_{\mathrm{k}}(\mathbf{k}',\mathrm{t})\hat{\mathrm{u}}_{\mathrm{l}}(\mathbf{k}'',\mathrm{t}) \tag{8}$$

where the effective viscosity (ν_e) is the sum of kinematic viscosity (ν_0) in Eq. 6 and renormalized viscosity ($\nu_{\mathrm{r}}(\mathrm{k_c})$). Note that $\nu_{\mathrm{r}}(\mathrm{k_c})$ appears in the left hand side of Eq. 8 as a result of the truncation of modes $\hat{\mathbf{u}}(\mathbf{k},t)$, $\hat{\mathbf{u}}(\mathbf{k}'',t)$ where $|\mathbf{k}'| > \mathrm{k_c}$, $|\mathbf{k}''| > \mathrm{k_c}$.
In Sec III we discuss different computational methodologies associated with our simulations.

III. SIMULATION DETAILS

We solve Eq. 3 for DNS and Eq. 8 for LES inside a periodic cubical box of size $2\pi\times2\pi\times2\pi$ using the pseudo-spectral code Tarang[18]. The allowed wavenumbers are $\mathrm{k_x} = (-\mathrm{n_x}/2 : \mathrm{n_x}/2)$, $\mathrm{k_y} = (-\mathrm{n_y}/2 : \mathrm{n_y}/2)$, $\mathrm{k_z} = (-\mathrm{n_z}/2 : \mathrm{n_z}/2)$. Owing to the conjugate symmetry for the Fourier modes $\hat{\mathbf{u}}(\mathbf{k}'',t) = \hat{\mathbf{u}}^*(-\mathbf{k},t)$, only half of the $\mathrm{k_z}$ modes $(0 : \mathrm{n_z}/2)$ are considered. In our simulations time-marching is done using second-order Runge-Kutta method. Furthermore, the 2/3 rule[19] is used for dealiasing and the Courant-Friedrichs-Lewy condition is used for determining the time step $\Delta\mathrm{t}$. In all the simulations a value of kinematic viscosity, $\nu_0 = 10^{-3}$, is used. Note that the total number of modes left after padding off one-third of the available modes, for dealiasing through the 2/3 rule, are given as $\frac{2}{3}\times\mathrm{N}$. For a sharp spectral cutoff, the cutoff wavenumber is the largest magnitude wavenumber in the low resolution grids and hence $\mathrm{k_c} = \frac{2}{3}\times\mathrm{N}/2 = \mathrm{N}/3$.
We have performed LES for three grid resolutions - 32^2, 64^3 and 128^3. Initial conditions for all the simulations were obtained by a spectral-reduction of available DNS 512^3 data, for fully developed turbulence at $\mathrm{R}_\lambda = 315$, to the required grid resolution. Starting with

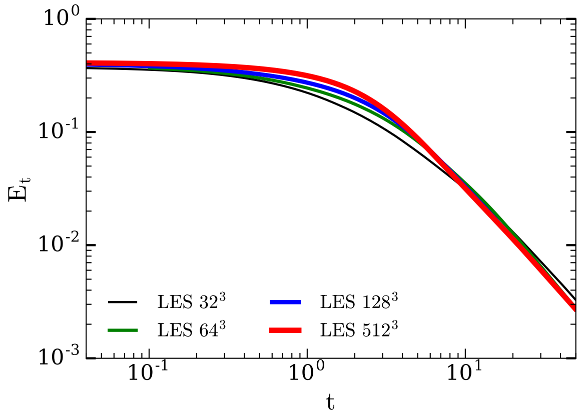

FIG. 1. Temporal evolution of turbulence kinetic energy. Note that the evolution is similar for LES and DNS.

these initial conditions, the turbulence is allowed to decay in DNS 512^3 as well as in LES for the three grid resolutions, till non dimensional time $t = 50$ when turbulence has decayed completely and the effective viscosity saturates to a constant value equal to that of kinematic viscosity.

In the following section we will compare the results obtained from the large eddy simulations against those obtained from direct numerical simulations.

IV. RESULTS AND DISCUSSIONS

In this section we compare the results obtained from LES and DNS using different Fourier and real space diagnostics. Note that in Fourier space, the kinetic energy spectrum $E_u(k)$ is the kinetic energy of the wavenumber shell of radius k and kinetic energy flux $\Pi_u(k_0)$ can be interpreted as the kinetic energy leaving a wavenumber sphere of radius k_0 and is given as[20],

$$\Pi_u(k_0) = \sum_{k \geq k_0} \sum_{p < k_0} \delta_{k,p+q} \mathrm{Im}[\mathbf{k}.\mathbf{u}(\mathbf{q})][\mathbf{u}^*(\mathbf{k}).\mathbf{u}(\mathbf{p})] \tag{9}$$

Fig. 1 shows the temporal evolution of turbulent Kinetic energy E_t for DNS and LES. Note that the initial energy for lower grid size simulations is small as compared to higher grid size simulations. This is because of the lesser number of modes in the former. Note that the evolution of turbulent kinetic energy in LES is similar to that of DNS in time. Fig. 2 shows

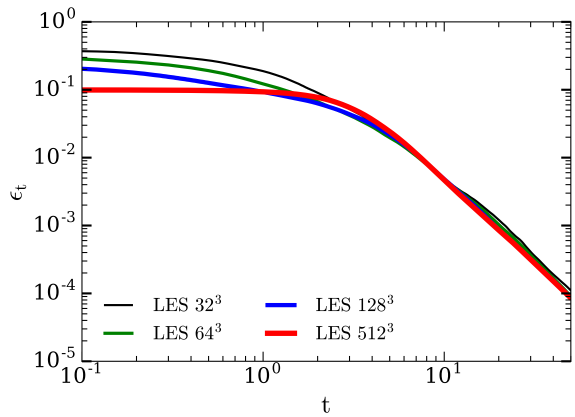

FIG. 2. Temporal evolution of Total dissipation ϵ_t. Note that ϵ_t increases with decreasing grid resolution.

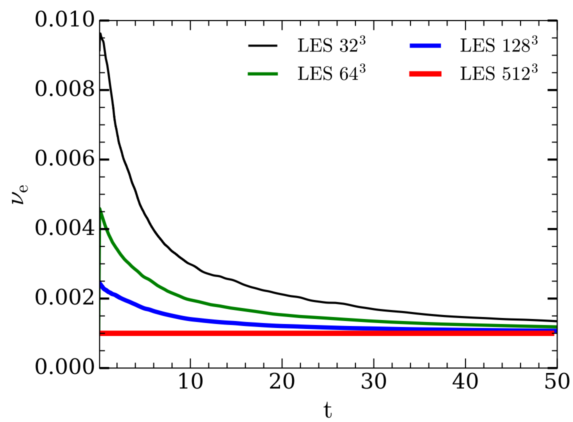

FIG. 3. Temporal evolution of effective viscosity $\nu_{\rm e} = \nu_0 + \nu_{\rm r}$. Note that ν_e increases with decreasing grid resolution.

the evolution of the total dissipation ϵ_t for different simulations. Note that ϵ_t is higher for low resolution grids. The total dissipation is given as,

$$\epsilon_{\rm t} = \sum_{\rm k} 2\nu_{\rm e}{\rm k}^2{\rm E}({\rm k}) \tag{10}$$

There will be a little decrease in ϵ_t owing to the truncation of modes beyond the cutoff in LES. The effective viscosity, however, depends on the renormalized viscosity which in turn is proportional to ${\rm k_c}^{-4/3}$ (Eq. 7) and hence increases with a decreasing grid resolution. Thus,

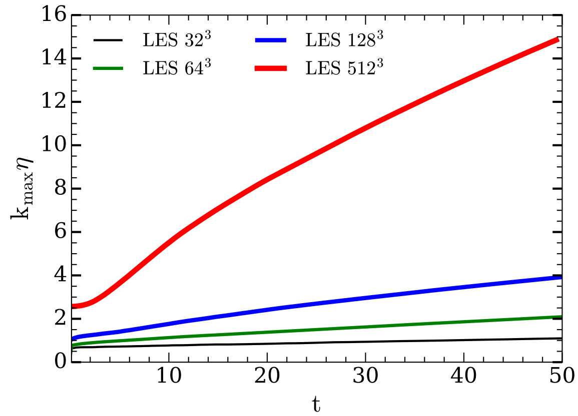

FIG. 4. Temporal evolution of $k_{max}\eta$. Note that $k_{max}\eta > 1$ for DNS as well as LES.

the total dissipation becomes higher for decreasing grid resolution.

Note from Fig. 4 that owing to an increase in the net viscosity (ν_e), there is an increase in the size of Kolmogorov's length scale η and hence LES brings about an increase in $k_{max}\eta$ such that it is greater than 1 for all the simulations.

Fig. 5 and Fig. 6 show the normalized kinetic energy (K.E) spectrum $E^*(k)$ and the kinetic energy flux $\Pi_u(k)$ respectively at non-dimensional time t=2. Note that there is a good match between the normalized K.E spectrum obtained from DNS and LES and that LES at all resolutions captures the expected Kolmogorov's $-5/3$[16] scaling in the inertial regime of wavenumbers very well. A small hump in the K.E spectrum at high wavenumbers for LES is due to the "bottleneck effect"[21] where the absence of sufficient number of modes impedes the transfer of turbulent kinetic energy to higher wavenumber modes and hence this energy accumulates near the cut-off. In Fig. 6, note the drop in $\Pi_u(k)$ for low resolution simulations (LES). This comes as an effect of increased total dissipation ϵ_t and reduced initial kinetic energy for the LES. Nevertheless, like DNS, the flux remains constant in the inertial range of wavenumbers for LES.

Fig. 7 and Fig. 8 show the velocity magnitude and finite-time-Lyapunov-exponent(FTLE) isosurfaces inside the computational domain for DNS 512^3 and LES 64^3. The isosurfaces for largest values of FTLE also identifies the dominant Lagrangian coherent structures (LCS) in the flow[22]. Note from the figures that, apart from sub-grid scale perturbations, which cannot be captured for a grid resolution as low as 64^3, the topology and the spatial distribution of

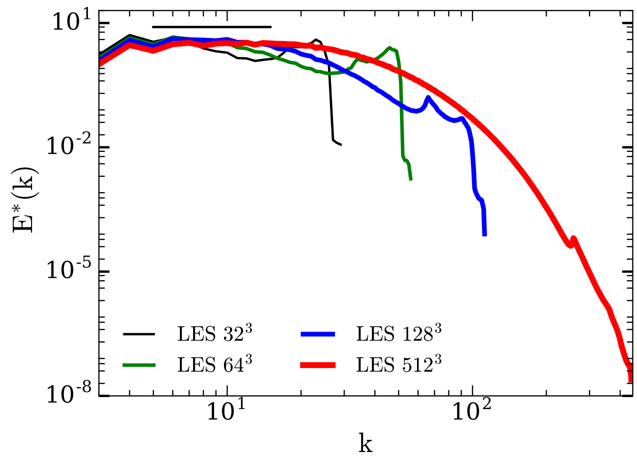

FIG. 5. Normalized Kinetic Energy $E^*(k) = E_u(k)k^{5/3}\Pi^{-2/3}$ Spectrum for DNS and LES at non-dimensional time t=2

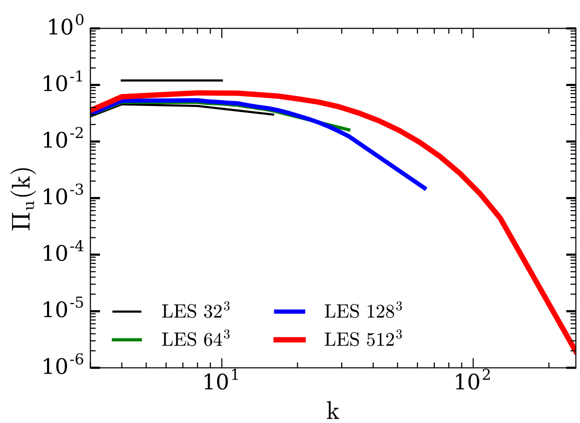

FIG. 6. Kinetic energy flux for DNS and LES at non-dimensional time t=2.

the large scale structures is predicted fairly well in LES when compared to that in DNS. This further confirms that the present renormalized viscosity based sub-grid scale model properly accounts for the effect of the truncated sub-grid modes on the large scale structures and predicts accurately, the dynamics of decaying turbulence. We remark here that the present renormalized viscosity is not suitable for computing anisotropic flows. Apart from this, the present eddy viscosity is entirely dissipative in nature and effects such as backscatter[23] are not accounted for. Anisotropic corrections similar to that in Yakhot and Orszag[11] can be

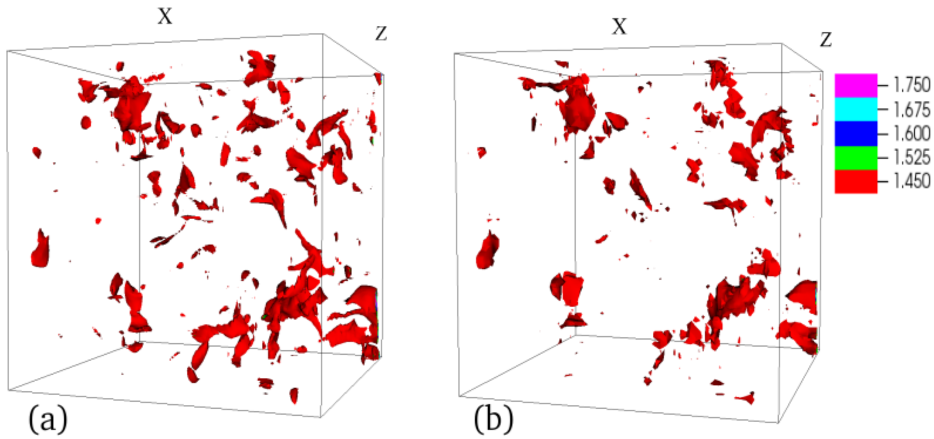

FIG. 7. Isosurfaces of the magnitude of velocity for (a) DNS 512^3 and (b) LES 64^3 at non-dimensional time t=2 for a range 1.45-1.75.

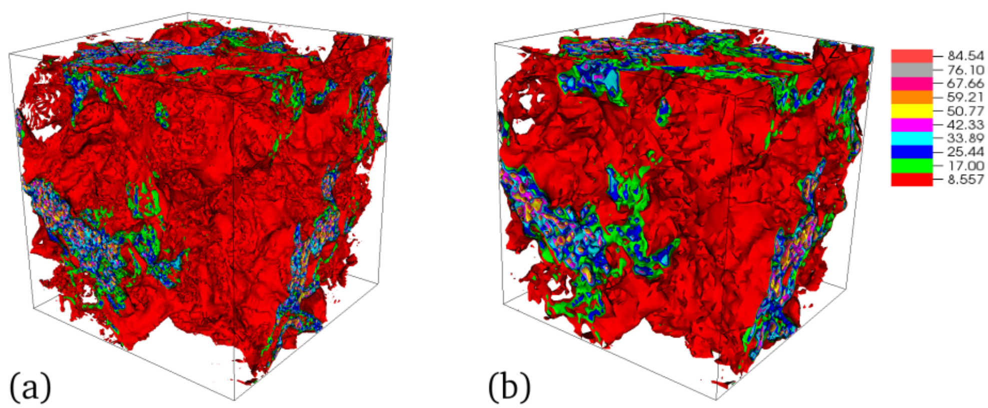

FIG. 8. Instantaneous Lagrangian coherent structures based on backward-time finite-time-Lyapunov-exponent (FTLE) field for (a) DNS 512^3 and (b) LES 64^3 at non-dimensional time t=2.

introduced for capturing anisotropy in flows. For capturing backscatter, a stochastic force that is uncorrelated in time can be used[24] along with the renormalized viscosity.

V. CONCLUSIONS

We have performed large eddy simulations of decaying homogeneous and isotropic turbulence in a periodic cubical box. LES was performed using renormalized viscosity derived using recursive renormalization-group scheme. The comparison of results for turbulent kinetic energy flux, kinetic energy spectrum and isosurfaces of the velocity magnitude and finite-time-Lyapunov-exponent show that the current formulation for renormalized viscosity, which is mathematically more consistent as compared to ϵ-expansion RNG based renormalized viscosity, faithfully captures the dynamics of decaying turbulent flows. Building on the present success in performing LES of decaying turbulence, development of sub-grid scale models based on the present formulation for renormalized viscosity and capable of accurately capturing anisotropy and backscatter in turbulent flows can be carried out.

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