

Energy and enstrophy spectra and fluxes for the inertial-dissipation range of two-dimensional turbulence

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In this paper, using Pao's conjecture [Y.-H. Pao, *Phys. Fluids* 11, 1371 (1968)], we derive expressions for the spectra and fluxes of kinetic energy and enstrophy for two-dimensional (2D) forced turbulence that extend beyond the inertial range. In these expressions, the fluxes and the spectra contain additional factors of the exponential form. To validate these model predictions, we perform numerical simulations of 2D turbulence with external force applied at $k = k_f$ in the intermediate range. The numerical results match with the model predictions, except for the energy and enstrophy fluxes for $k < k_f$, where the fluxes exhibit significant fluctuations. We show that these fluctuations arise due to the unsteady nature of the flow at small wavenumbers. For the $k < k_f$, the shell-to-shell energy transfers computed using numerical data show forward energy transfers among the neighbouring shells, but backward energy transfers for other shells.

I. INTRODUCTION

Turbulence is an omnipresent phenomena.¹⁻⁴ Though many natural and laboratory flows are three-dimensional, many astrophysical and geophysical flows exhibit two-dimensional (2D) or quasi-two-dimensional behavior.⁵⁻¹⁰ For example, strong rotation suppresses the velocity component in the direction of rotation.¹¹⁻¹³ Similarly, a strong external magnetic field in magnetohydrodynamics,¹⁴⁻¹⁷ and strong gravity in planetary environments¹⁸⁻²⁰ make the flow quasi-two-dimensional. Therefore, a good understanding of 2D turbulence is important for modeling such flows. In this paper, we address the spectra and fluxes of energy and enstrophy for the inertial-dissipation range of 2D turbulence.

Using analytical arguments, Kraichnan²¹ predicted a dual cascade for 2D turbulence that is forced at an intermediate scale ($k \approx k_f$). He showed an inverse cascade of kinetic energy for $k < k_f$, and forward cascade of enstrophy for $k > k_f$. In the corresponding regimes, the kinetic energy spectra are $E_u(k) = C\epsilon^{2/3}k^{-5/3}$ and $E_u(k) = C'\eta^{2/3}k^{-3}$ respectively; here ϵ , η are respectively the energy and enstrophy dissipation rates (or injection rates), and C, C' are constants. Further, Kraichnan²² derived a logarithmic correction to the latter spectrum.

The aforementioned inverse energy cascade and double cascade in energy spectrum have been observed in many laboratory experiments, for example by Paret and Tabeling²³, Rutgers,²⁴ and Kellay et. al.²⁵ In numerical simulations, the same phenomena has also been observed by Siggia and Aref²⁶, Frisch and Sulem²⁷, and Borue.²⁸ For a forced 2D turbulence, the large-scale energy grows in time.²⁹ In the regime with forward enstrophy transfer,

the energy spectrum is typically steeper than k^{-3} , both in numerical simulations³⁰ and in experiments.³¹ Moreover, Scott³², Fontane *et al.*³³ report some deviations from the theoretical predictions of Kraichnan²¹. Pandit *et al.*³⁴ describe properties of 2D flows in the presence of complex forces.

Boffetta³⁵ performed direct numerical simulations of forced 2D Navier-Stokes equations and studied the energy and enstrophy cascade regimes with good accuracy. Boffetta³⁵ employed Ekman friction to suppress energy growth at large scales. Besides the above spectral laws, variable energy flux, irregular and non-local energy transfer have also been studied for 2D turbulence.³⁶ However, despite many years of work, there are some discrepancies on the scaling laws. For example, in the $k > k_f$ regime, the enstrophy spectrum is typically steeper than k^{-3} .

Kolmogorov's theory^{37,38} yields $k^{-5/3}$ energy spectrum for 3D hydrodynamic turbulence. Pao³⁹ generalised this scaling to inertial-dissipation range by postulating that the ratio of the energy spectrum and energy flux is independent of the kinematic viscosity, and that it depends on the dissipation rate and local wavenumber. We employ Pao's conjecture³⁹ to 2D turbulence, and extend the $k^{-5/3}$ and k^{-3} spectra and corresponding fluxes predicted by Kraichnan²¹ beyond the inertial range.

We simulate 2D turbulence numerically and compute the spectra and fluxes of energy and enstrophy, and compare the numerical results with the predictions of extended model of spectra and fluxes based on Pao's conjecture. We observe good agreement between the numerical and model results for $k > k_f$. However, they differ for $k < k_f$ possibly due to the unsteady nature of 2D turbulence.

The present paper is structured as follows. In Sec. II, we describe the governing equations for a forced two-dimensional incompressible fluid. In Sec. III we derive the spectra and fluxes of energy and enstrophy using Pao's conjecture. In Sec. IV, we describe our numerical

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procedure. Sec. V contains simulation results and comparison with model predictions. We conclude in Sec. VI.

II. GOVERNING EQUATIONS

The Navier-Stokes equations for a forced two-dimensional incompressible fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}_u, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

where \mathbf{u} and p are the velocity and pressure fields respectively, ν is the kinematic viscosity, and \mathbf{F}_u is the external force. We take density to be constant ($\rho = 1$). The flow is two-dimensional in xy plane, and the vorticity is a scalar: $\omega = (\nabla \times \mathbf{u}) \cdot \hat{z}$. Taking a curl of Eq. (1) yields the following dynamical equation for the vorticity field:

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega + F_\omega, \quad (3)$$

where $F_\omega = [\nabla \times \mathbf{F}_u]_z$.

For an inviscid and force-free regime of 2D hydrodynamic flow, the total kinetic energy (KE), E_u , and the total enstrophy E_ω , defined below

$$E_u = \int d\mathbf{r} u^2(\mathbf{r})/2; \quad E_\omega = \int d\mathbf{r} \omega^2(\mathbf{r})/2 \quad (4)$$

are conserved. These quadratic invariants play an important role in 2D turbulence.

The above equations for the velocity and vorticity fields are written in Fourier space as

$$\frac{d}{dt} \mathbf{u}(\mathbf{k}) + \mathbf{N}_u(\mathbf{k}) = -i\mathbf{k}p(\mathbf{k}) + \mathbf{F}_u(\mathbf{k}) - \nu k^2 \mathbf{u}(\mathbf{k}), \quad (5)$$

$$\frac{d}{dt} \omega(\mathbf{k}) + N_\omega(\mathbf{k}) = F_\omega(\mathbf{k}) - \nu k^2 \omega(\mathbf{k}), \quad (6)$$

$$\mathbf{k} \cdot \mathbf{u}(\mathbf{k}) = 0, \quad (7)$$

where

$$\mathbf{N}_u(\mathbf{k}) = i \sum_{\mathbf{p}} \{\mathbf{k} \cdot \mathbf{u}(\mathbf{q})\} \mathbf{u}(\mathbf{p}), \quad (8)$$

$$N_\omega(\mathbf{k}) = i \sum_{\mathbf{p}} \{\mathbf{k} \cdot \mathbf{u}(\mathbf{q})\} \omega(\mathbf{p}), \quad (9)$$

with $\mathbf{q} = \mathbf{k} - \mathbf{p}$. Note that the pressure $p(\mathbf{k})$ is derived by taking dot product of Eq. (5) with $i\mathbf{k}$ and by employing $\mathbf{k} \cdot \mathbf{u}(\mathbf{k}) = 0$:

$$p(\mathbf{k}) = \frac{i}{k^2} \mathbf{k} \cdot \{\mathbf{N}_u(\mathbf{k}) - \mathbf{F}_u(\mathbf{k})\}. \quad (10)$$

To derive a dynamical equation for the modal KE $E_u(\mathbf{k}) = |\mathbf{u}(\mathbf{k})|^2/2$ and modal enstrophy $E_\omega(\mathbf{k}) = |\omega(\mathbf{k})|^2/2$, we perform dot products of Eq. (5) with $\mathbf{u}^*(\mathbf{k})$,

and Eq. (6) with $\omega^*(\mathbf{k})$, and add the resultant equations with their complex conjugates. These operations yield

$$\begin{aligned} \frac{d}{dt} E_u(\mathbf{k}) &= \sum_{\mathbf{p}} \Im [\{\mathbf{k} \cdot \mathbf{u}(\mathbf{q})\} \{\mathbf{u}(\mathbf{p}) \cdot \mathbf{u}^*(\mathbf{k})\}] \\ &\quad + \Re [\mathbf{F}_u(\mathbf{k}) \cdot \mathbf{u}^*(\mathbf{k})] - 2\nu k^2 E_u(\mathbf{k}) \\ &= T_u(\mathbf{k}) + \mathcal{F}_u(\mathbf{k}) - D_u(\mathbf{k}) \end{aligned} \quad (11)$$

and

$$\begin{aligned} \frac{d}{dt} E_\omega(\mathbf{k}) &= \sum_{\mathbf{p}} \Im [\{\mathbf{k} \cdot \mathbf{u}(\mathbf{q})\} \{\omega(\mathbf{p}) \omega^*(\mathbf{k})\}] \\ &\quad + \Re [F_\omega(\mathbf{k}) \omega^*(\mathbf{k})] - 2\nu k^2 E_\omega(\mathbf{k}) \\ &= T_\omega(\mathbf{k}) + \mathcal{F}_\omega(\mathbf{k}) - D_\omega(\mathbf{k}), \end{aligned} \quad (12)$$

where $\Re[\cdot]$, $\Im[\cdot]$ are real and imaginary parts of the argument respectively, $T_u(\mathbf{k})$, $T_\omega(\mathbf{k})$ are respectively the rate of KE and enstrophy transfers to the modal KE and modal enstrophy by nonlinearity, $\mathcal{F}_u(\mathbf{k})$, $\mathcal{F}_\omega(\mathbf{k})$ are respectively the modal KE and enstrophy injection rates by the external force, and $D_u(\mathbf{k})$, $D_\omega(\mathbf{k})$ are respectively the dissipation rates of the modal KE and enstrophy. We define KE and enstrophy fluxes for a wavenumber sphere of radius k_0 as

$$\Pi_u(k_0) = - \int_0^{k_0} T_u(k) dk, \quad (13)$$

$$\Pi_\omega(k_0) = - \int_0^{k_0} T_\omega(k) dk. \quad (14)$$

Using the above equations we can derive the following equations for one-dimensional spectra^{4,20}

$$\frac{\partial}{\partial t} E_u(k, t) = - \frac{\partial}{\partial k} \Pi_u(k, t) + \mathcal{F}_u(k, t) - D_u(k, t), \quad (15)$$

$$\frac{\partial}{\partial t} E_\omega(k, t) = - \frac{\partial}{\partial k} \Pi_\omega(k, t) + \mathcal{F}_\omega(k, t) - D_\omega(k, t). \quad (16)$$

We employ the above equations to derive spectra and fluxes for the KE and enstrophy in the inertial range and beyond.

III. THE SPECTRA AND FLUXES OF KINETIC ENERGY AND ENSTROPY BEYOND THE INERTIAL RANGE

In this section we extend Kraichnan²¹'s formulas for the KE and enstrophy spectra and fluxes beyond the inertial range using Pao's conjecture.

Under a steady state, in Eqs. (15, 16), we set $\partial/\partial t = 0$. In addition, in the inertial range, the injection rates by the external force vanish, while the dissipation rates are negligible. Hence,

$$\frac{d}{dk} \Pi_u(k) = 0; \quad \frac{d}{dk} \Pi_\omega(k) = 0 \quad (17)$$

that leads to constancy of KE and enstrophy fluxes. Kraichnan²¹ showed that $\Pi_u(k)$ is constant for $k < k_f$, while $\Pi_\omega(k)$ is constant for $k > k_f$. For these regimes, dimensional analysis yields²¹

$$E_u(k) = C\epsilon^{2/3}k^{-5/3} \quad \text{for } k < k_f \quad (18)$$

$$E_u(k) = C'\eta^{2/3}k^{-3} \quad \text{for } k > k_f \quad (19)$$

where ϵ, η are respectively the KE and enstrophy injection rates, and C, C' are constants.

To extend the above scaling beyond inertial range, but still away from the forcing range, we retain $D_u(k)$ and $D_\omega(k)$ in Eq. (17) that yields

$$\frac{d}{dk}\Pi_u(k) = -2\nu k^2 E_u(k), \quad (20)$$

$$\frac{d}{dk}\Pi_\omega(k) = -2\nu k^2 E_\omega(k). \quad (21)$$

The above relations are valid for all wavenumbers. The above two equations have four unknowns, hence they cannot be uniquely solved. To overcome this difficulty, we extend Pao's conjecture³⁹ for 3D hydrodynamic turbulence to 2D turbulence. We assume that for $k < k_f$, $E_u(k)/\Pi_u(k)$ is a function of ϵ and k , while for $k > k_f$, $E_u(k)/\Pi_u(k)$ is a function of η and k . Under these assumptions, dimensional analysis yields

$$\frac{E_u(k)}{\Pi_u(k)} = -C\epsilon^{-1/3}k^{-5/3} \quad \text{for } k < k_f, \quad (22)$$

$$\frac{E_\omega(k)}{\Pi_\omega(k)} = C'\eta^{-1/3}k^{-3} \quad \text{for } k > k_f. \quad (23)$$

Note that the negative sign in Eq. (22) is due to the fact that $\Pi_u(k) < 0$.

Substitution of Eqs. (22, 23) in Eqs. (20, 21) respectively yields

$$\frac{d}{dk}\Pi_u(k) = 2C\nu\epsilon^{-1/3}k^{1/3}\Pi_u(k) \quad \text{for } k < k_f, \quad (24)$$

$$\frac{d}{dk}\Pi_\omega(k) = -2C'\nu\eta^{-1/3}k^{-1}\Pi_\omega(k) \quad \text{for } k > k_f. \quad (25)$$

For $k < k_f$ regime, the solution of Eq. (24) is

$$\Pi_u(k) = -\epsilon_u \exp\left(\frac{3}{2}C(k/k_d)^{4/3}\right), \quad (26)$$

$$E_u(k) = C\epsilon_u^{2/3}k^{-5/3} \exp\left(\frac{3}{2}C(k/k_d)^{4/3}\right), \quad (27)$$

where $k_d = (\frac{\epsilon_u}{\nu^3})^{1/4}$. For $k > k_f$, the solution of Eq. (25) is

$$\Pi_\omega(k) = \epsilon_\omega \exp(-C'(k/k_{d2D})^2), \quad (28)$$

$$E_\omega(k) = C'\epsilon_\omega^{2/3}k^{-1} \exp(-C'(k/k_{d2D})^2), \quad (29)$$

$$E_u(k) = C'\epsilon_\omega^{2/3}k^{-3} \exp(-C'(k/k_{d2D})^2), \quad (30)$$

where $k_{d2D} = \frac{\epsilon_\omega^{1/6}}{\sqrt{\nu}}$ is the enstrophy dissipation wavenumber. Note that $E_u(k)$ of Eq. (30) is steeper than k^{-3} in

the inertial-dissipation range. In Sec. V we show consistency of the above steepening with the numerical results.

A question is, what is the nature of enstrophy flux for $k < k_f$, and energy flux for $k > k_f$? Boffetta³⁵, and Boffetta and Ecke¹⁰ conjectured that $\Pi_\omega(k) \approx 0$ for $k < k_f$, and $\Pi_u(k) \approx 0$ for $k > k_f$. But this is not the case because Eqs. (20, 21) yields

$$\frac{d\Pi_\omega(k)}{d\Pi_u(k)} = \frac{1}{k^2}. \quad (31)$$

In fact, we can determine these fluxes using Eqs. (20, 21) in the following manner.

For $k < k_f$, substitution of $E_u(k)$ of Eq. (26) in Eq. (21) yields

$$\begin{aligned} \Pi_\omega(k) &= -2\nu \int^k k'^4 E_u(k') dk' \\ &= -2\nu C\epsilon_u^{2/3} \int^k k'^{7/3} \exp\left(\frac{3}{2}C(k'/k_d)^{4/3}\right) dk' \\ &= -2\nu C\epsilon_u^{2/3} k_d^{10/3} \int^{x'} dx' x'^{7/3} \exp\left(\frac{3}{2}Cx'^{4/3}\right), \end{aligned} \quad (32)$$

where $x' = k/k_d$. Similarly, for $k > k_f$, substitution of $E_u(k)$ of Eq. (30) in Eq. (20) yields

$$\begin{aligned} \Pi_u(k) &= -2\nu \int^k k'^2 E_u(k') dk' \\ &= -\frac{\epsilon_\omega}{k_{d2D}^2} \int^k \frac{1}{k'} \exp(-C'(k'/k_{d2D})^2) dk' \end{aligned} \quad (33)$$

$$= -\frac{\epsilon_\omega}{k_{d2D}^2} \text{Ei}(-C'(k/k_{d2D})^2), \quad (34)$$

where Ei is the exponential integral⁴⁰. Asymptotically, $-\text{Ei}(-x) \sim \exp(-x)/x$. Hence,

$$\Pi_u(k) \approx \frac{\epsilon_\omega}{k^2} \exp(-C'(k/k_{d2D})^2). \quad (35)$$

Note however that the aforementioned model of spectra and fluxes of KE and enstrophy assume steady state. As we show in Sec. V, this assumption does not hold due to unsteady nature of 2D turbulence. It has been reported that the large-scale KE grows with time. As a result, some of the above predictions match with the simulation results, while some do not.

We will attempt to verify the above scaling functions using numerical simulations.

IV. NUMERICAL PROCEDURE

In the present paper, we perform numerical simulations of forced 2D hydrodynamic turbulence using spectral method. The system is doubly-periodic in a domain of size $2\pi \times 2\pi$. We employ two different grid resolutions 2048² and 8192² to make sure that our results are

grid independent. The equations are solved using a fully dealiased, parallel pseudo-spectral code TARANG⁴¹ with fourth-order Runge-Kutta time marching scheme. For dealiasing purpose, 3/2-rule is chosen.^{42,43} The viscosity for the 2048² and 8192² grids are set at 1×10^{-3} and 3×10^{-4} respectively that yields corresponding Reynolds numbers of 1.2×10^4 and 4.2×10^5 . For our simulation we employ eddy turnover as unit of time. The simulations for the two grids were run up to $t_{\text{final}} = 10.0$ and 1.74 respectively. We employ Courant-Friedrichs-Lewy (CFL) condition to determine the timestep dt . For the two grids, the average time steps are 3.4×10^{-5} and 6.9×10^{-5} respectively.

We force the flow at wavenumber band $k_f = (50, 51)$ and $(100, 101)$ for 2048² and 8192² grids respectively. These resolutions provide more than a decade of inverse cascade regime. The enstrophy cascades forward in $k > k_f$ regime, but the spectrum is steeper than k^{-3} due to the dissipation effects.

Using the numerical data we compute the one-dimensional energy and enstrophy spectrum using

$$E_u(k) = \frac{1}{2} \sum_{k-1 < |\mathbf{k}'| \leq k} |\mathbf{u}(\mathbf{k}')|^2, \quad (36)$$

$$E_\omega(k) = \frac{1}{2} \sum_{k-1 < |\mathbf{k}'| \leq k} |\boldsymbol{\omega}(\mathbf{k}')|^2. \quad (37)$$

The energy flux $\Pi_u(k_0)$ is defined as the energy leaving the sphere of radius k_0 due to nonlinearity. This quantity is computed as^{44,45}

$$\Pi_u(k_0) = \sum_{|\mathbf{p}| \leq k_0} \sum_{|\mathbf{k}| > k_0} S^{uu}(\mathbf{k}|\mathbf{p}|\mathbf{q}) \quad (38)$$

where

$$S^{uu}(\mathbf{k}|\mathbf{p}|\mathbf{q}) = \Im \{ [\{\mathbf{k} \cdot \mathbf{u}(\mathbf{q})\} \{\mathbf{u}(\mathbf{p}) \cdot \mathbf{u}^*(\mathbf{k})\}] \}, \quad (39)$$

is the *node-to-mode* energy transfer from Fourier mode $\mathbf{u}(\mathbf{p})$ to Fourier mode $\mathbf{u}(\mathbf{k})$ with Fourier mode $\mathbf{u}(\mathbf{q})$ acting as a mediator. Note that $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ form a triad with $\mathbf{k} = \mathbf{p} + \mathbf{q}$. Similarly, we compute the enstrophy flux as

$$\Pi_\omega(k_0) = \sum_{|\mathbf{p}| \leq k_0} \sum_{|\mathbf{k}| > k_0} S^{\omega\omega}(\mathbf{k}|\mathbf{p}|\mathbf{q}) \quad (40)$$

where

$$S^{\omega\omega}(\mathbf{k}|\mathbf{p}|\mathbf{q}) = \Im \{ [\{\mathbf{k} \cdot \mathbf{u}(\mathbf{q})\} \{\boldsymbol{\omega}(\mathbf{p}) \cdot \boldsymbol{\omega}^*(\mathbf{k})\}] \}, \quad (41)$$

is the *node-to-mode* energy transfer from Fourier mode $\boldsymbol{\omega}(\mathbf{p})$ to Fourier mode $\boldsymbol{\omega}(\mathbf{k})$ with Fourier mode $\mathbf{u}(\mathbf{q})$ acting as a mediator.

The energy and enstrophy fluxes provide insights into the global transfers in the system. For a more detailed picture, we compute the shell-to-shell energy transfers. We divide the wavenumber space into various concentric shells. The shell-to-shell energy transfer from shell m to shell n is given by

$$T_{u,n}^{u,m} = \sum_{\mathbf{p} \in m} \sum_{\mathbf{k} \in n} S^{uu}(\mathbf{k}|\mathbf{p}|\mathbf{q}). \quad (42)$$

In the present work, we compute the shell-to-shell energy transfer for $k < k_f$ regime. We compare the our numerical result with those for 3D hydrodynamic turbulence for which the energy transfer is local and forward in the inertial range, that is, the dominant energy transfer is from shell m is to $m + 1$. For better resolution, we perform the shell-to-shell transfer computations for 8192³ grid with log-binned shells. We divide the Fourier space into 20 concentric shells; the inner and outer radii of the i^{th} shell are k_{i-1} and k_i respectively. The shell radii for $N = 8192^2$ grids are $k_i = \{0, 2, 4, 8, 8 \times 2^{s(i-3)}, \dots, 2048, 4096\}$, where $s = 8/15$ and i is the shell index. Inertial range shells have been chosen by logarithmic binning because of the power law physics in the inertial range.

In this paper we do not report the shell-to-shell enstrophy transfer in the $k > k_f$ regime due to lack of constant enstrophy. In future we plan to perform simulations with $k_f \approx 1$ that would provide significant wavenumber regime of constant enstrophy flux. It will be meaningful to perform shell-to-shell enstrophy transfer computations using such data.

In the next section we will report numerical results on the spectra and fluxes of energy and enstrophy.

V. RESULTS AND DISCUSSIONS

In the present section we will report the numerically computed spectra and fluxes of energy and enstrophy. We will compare these results with the model predictions of Sec. III.

A. Energy spectra

In this subsection, we describe the energy and enstrophy spectra of 2D turbulence. In Fig. 1(a,b) we plot these spectra for 2048² and 8192² grids. The numerically computed spectra are exhibited using red solid curves, and the model predictions of Sec. III using dashed black curves. In this figure we also plot the predictions of Kraichnan's theory using solid black lines. We observe that the model predictions match with the numerical results quite well.

For $k < k_f$, the numerical results and model predictions yield $E_u(k) \sim k^{-5/3}$, which is the prediction of Kraichnan²¹. However, for $k > k_f$, $E_u(k)$ is steeper than Kraichnan²¹'s predictions of k^{-3} . The steepening of E_u compared to k^{-3} is due to the dissipative effects. Interestingly, as shown in the figure, the dissipative effects are nicely captured by the model equation (30). We believe that the model of Sec. III could explain the steepening of $E_u(k)$ reported by Legras *et al.*³⁰ and Kellay *et al.*³¹ for the $k > k_f$ regime. Note that $E_\omega(k) = E_u(k)/k^2$, hence they are not reported separately.

In the next subsection we will describe the energy and enstrophy fluxes computed using numerical data, as well

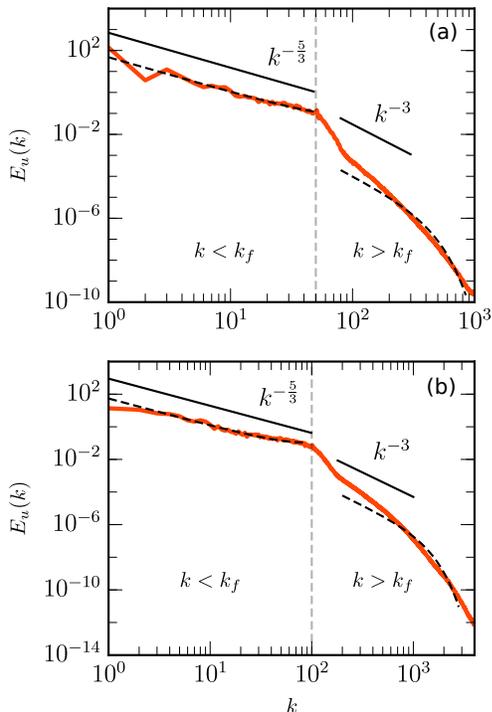


FIG. 1. Kinetic energy spectrum for 2D forced turbulence: (a) using 2048^2 grid simulation with forcing at $k_f = (50, 51)$, (b) using 2048^2 grid simulation with forcing at $k_f = (100, 101)$. The forcing wavenumbers are indicated by dashed vertical lines. The plots exhibit numerical results (solid red curves), model predictions [Eqs. (27, 30)] (dashed black curves) and Kraichnan²¹'s predictions (solid black lines). The model predictions match with the numerical results quite well.

as those predicted by the model of Sec. III.

B. Energy and enstrophy fluxes

We compute the energy and enstrophy fluxes for 2D turbulence using Eqs. (38, 40). We compute these quantities for both the grid resolutions (2048^2 and 8192^2). The numerically computed fluxes are exhibited in Fig. 2. The left and right panels of Fig. 2 illustrate the energy and enstrophy fluxes respectively.

The top panel of Fig. 2 exhibits the energy and enstrophy fluxes for a single time frame of 2048^2 grid run. We observe that these fluxes exhibit significant fluctuations for $k < k_f$. Therefore, we compute average fluxes. The fluxes in the middle and bottom panels are computed by averaging over 700 and 35 different time frames for 2048^2 and 8192^2 grids respectively. In the plots the red curves represent the numerically computed fluxes, while the black dashed lines represent the model predictions of Sec. III.

A careful observation of the figure shows that for

$k > k_f$, the model predictions of the KE and enstrophy fluxes, Eqs. (28, 34), are in good agreement with the numerical results. Thus, Eqs. (28, 34) describe 2D turbulence satisfactorily for the $k > k_f$ regime.

However, for $k < k_f$, the model predictions fail to describe from the numerical results well. In Fig. 2(a,b), the energy and enstrophy fluxes computed using a single frame data exhibits significant fluctuations. The fluctuations are somewhat suppressed on averaging, as shown in Fig. 2(c-f), yet the enstrophy flux for 8192^2 grid shows large fluctuations. We believe that the fluctuations in the fluxes are due to the unsteady nature of the flow.

Two-dimensional turbulence exhibits inverse cascade of kinetic energy that leads to formation of large-scale structures. The viscous dissipation is weak for small wavenumbers, hence the energy feed at the large-scale structures make the flow unsteady. As a result, Eqs. (20, 21) are not valid for $k < k_f$ regime. To quantify the unsteadiness of the flow, we compute all the terms of Eq. (15) for 2048^2 grid and compare them. In Fig. 3 we plot $|\partial E(k)/\partial t|$ and $|-d\Pi_u(k)/dk - D_u(k)|$ with respect to k . Though the left-hand and right-hand sides of Eq.(15) match with each other, noticeably, $|\partial E(k)/\partial t|$ is significant for $k < k_f$. Note however that these quantities are small for $k > k_f$. This is the reason for the unsteady nature of the flow that leads to strong fluctuations in $\Pi_u(k)$ and $\Pi_\omega(k)$ in the $k < k_f$ regime.

In the next subsection, we describe the shell-to-shell energy transfers for 2D turbulence.

C. Shell to shell energy transfers

In the present subsection we describe the shell-to-shell energy transfers for 2D turbulence. We compare our results with three-dimensional turbulence for which the shell-to-shell transfers are local and forward^{46,47}.

We compute the shell-to-shell energy transfers in the wavenumber band $k < k_f$ using the formula of Eq. (42). As described in Sec. IV, for 8192^2 grid simulation we divide this wavenumber region into 20 shells. The computed transfers are exhibited in Figures 4 and 5.

In Fig. 4(a), we plot the shell-to-shell energy transfers $T_{u,n}^{u,m}$ vs. $n - m$ computed for a single frame. Here m, n are the giver and receiver shells respectively. These transfers exhibit significant fluctuations for different data sets, hence we average the transfer rates for 35 frames. The averaged transfers are exhibited in Fig. 4(b). As shown in the figures, specially Fig. 4(b), shell n receives energy from shell $n - 1$ and gives energy to shell $n + 1$. Hence, among the nearest neighbour shells in the inertial range of $k < k_f$, the energy transfer in 2D hydrodynamic turbulence is forward. Note however that $T_{u,n}^{u,m} < 0$ when $n - m > 2$ or 3 (for some shell). This implies that shell n receives energy from far away shells. Therefore, in 2D hydrodynamic turbulence, the shell-to-shell energy transfers to the neighboring shells are forward, but they are backward for the distant shells.

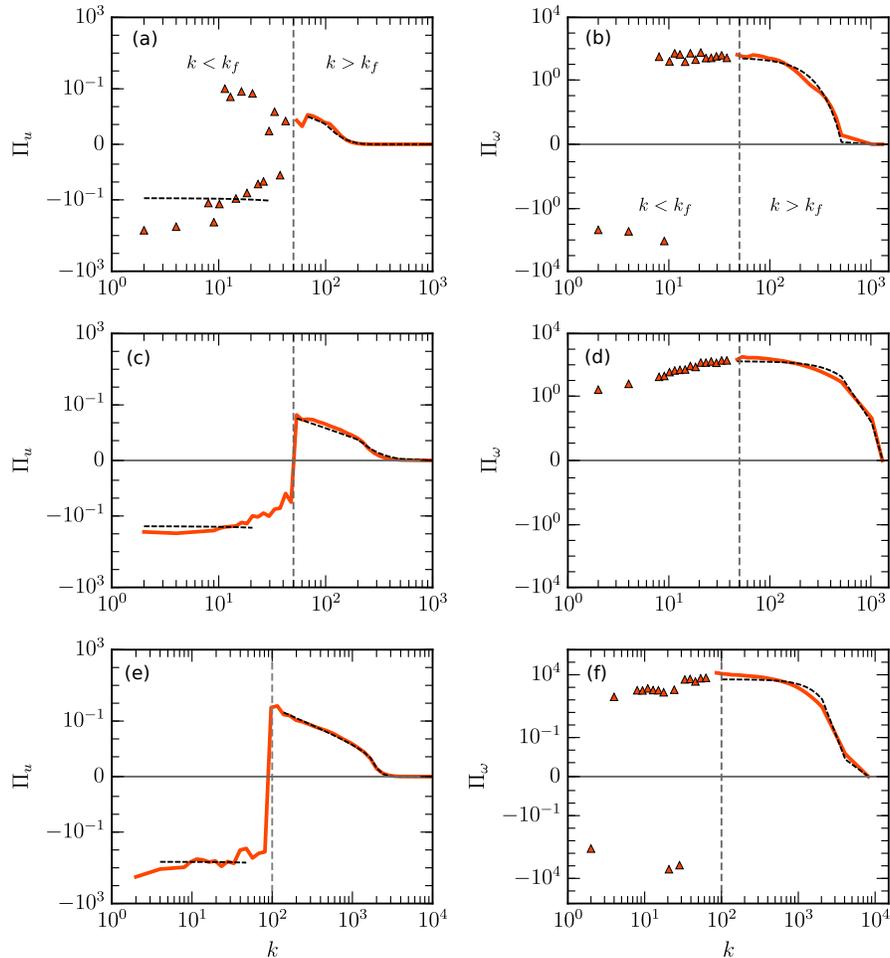


FIG. 2. Plot of kinetic energy flux, $\Pi_u(k)$ (left panel), and enstrophy fluxes, $\Pi_\omega(k)$ (right panel): (a,b) for a single time frame of 2048^2 run. (c,d) Time averaged $\Pi_u(k)$ and $\Pi_\omega(k)$ over 700 frames of 2048^2 run. (e, f) Time averaged $\Pi_u(k)$ and $\Pi_\omega(k)$ over 35 frames of 8192^2 run. The numerical results (red solid lines) match with the model predictions [Eqs. (28, 34)] (black dashed lines) in the $k > k_f$ regime. The discrepancies in the $k < k_f$ regime are due to the unsteady nature of the flow.

In Fig. 5, we exhibit the corresponding density plots— Fig. 5(a) for a single frame, and Fig. 5(b) the time averaged transfers. Here the indices of the x, y axes represent the receiver and giver shells respectively. Though the plot exhibit significant fluctuations, we can conclude that on an average, the shell-to-shell energy transfers for the nearest neighbours are forward, but those for the other neighbours are backward. These results are consistent with the results of Fig. 4.

The aforementioned shell-to-shell energy transfers of 2D turbulence differ significantly from the those of 3D turbulence for which the transfers are local and forward for the inertial range shells. This divergence between the 2D and 3D flows is due to the inverse cascade of energy. Verma *et al.*⁴⁷ computed the shell-to-shell energy transfers for 2D turbulence using field-theoretic tools and reported local forward and nonlocal backward energy transfers for $k < k_f$ (consistent with the aforementioned

numerical simulations); they showed that these complex transfers add up to yield a negative $\Pi_u(k)$. Here, the non-local backward energy transfers from many shells play a critical role.

We summarise our results in the next section.

VI. CONCLUSIONS

In this paper we present several results on 2D forced turbulence (with forcing employed at intermediate scales). Using Pao's conjecture, we extend Kraichnan²¹'s power law predictions for 2D turbulence beyond the inertial range. In the new scaling solution, the power laws are coupled with exponential functions of k .

To test the model predictions, we perform numerical solution of 2D turbulence on 2048^2 and 8192^2 grids with forcing at (50,51) and (100,101) wavenumber bands re-

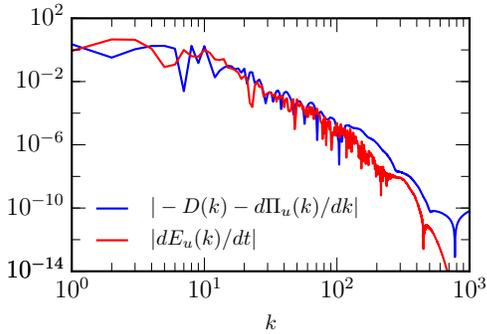


FIG. 3. For the 2048^2 grid simulation, plots of $| -D(k) - d\Pi_u(k)/dk |$ and $|\partial E(k, t)/\partial t|$. Significant measures of $|\partial E(k, t)/\partial t|$ indicates unsteady nature of the flow.

spectively. We compute the spectra and fluxes of energy and enstrophy using the numerical data, and compare them with the model predictions. We observe that the model predictions and numerical results on the spectra and fluxes of energy and enstrophy agree with each other for $k > k_f$. In this regime, the energy spectrum is steeper than k^{-3} , which is primarily due to the exponential factor $\exp(-(k/k_{d2D})^2)$ of Eq. (30). The situation however is different for $k < k_f$. Though the energy spectrum follows $k^{-5/3}$ power law, the energy and enstrophy fluxes exhibit significant fluctuations. We show that these fluctuations arise due to the unsteady nature of the flow and inverse energy cascade of kinetic energy. The fluctuations are somewhat suppressed on averaging. These issues need further investigation.

We also compute the shell-to-shell energy transfers in the $k < k_f$ regime. We observe forward energy transfers for the nearest neighbour shells, but backward energy transfers for the other shells, consistent with the analytical findings of Verma *et al.*⁴⁷. The nonlocal backward transfers add up to yield a negative energy flux. In addition, we observe that the shell-to-shell energy transfers in $k < k_f$ regime exhibits significant fluctuations among different frames due to the unsteady nature of the flow.

In summary, our findings on 2D turbulence sheds interesting light on the energy and enstrophy transfers. In future, we plan to extend the present work to the extended regime of constant enstrophy flux, in particular study the shell-to-shell enstrophy transfers and explore if they are local or nonlocal.

ACKNOWLEDGMENTS

We thank Manohar Sharma for useful discussions and Shaswant Bhattacharya for comments on the manuscript. Our numerical simulations were performed on Shaheen II at KAUST supercomputing laboratory, Saudi Arabia, under the project k1052. This work was sup-

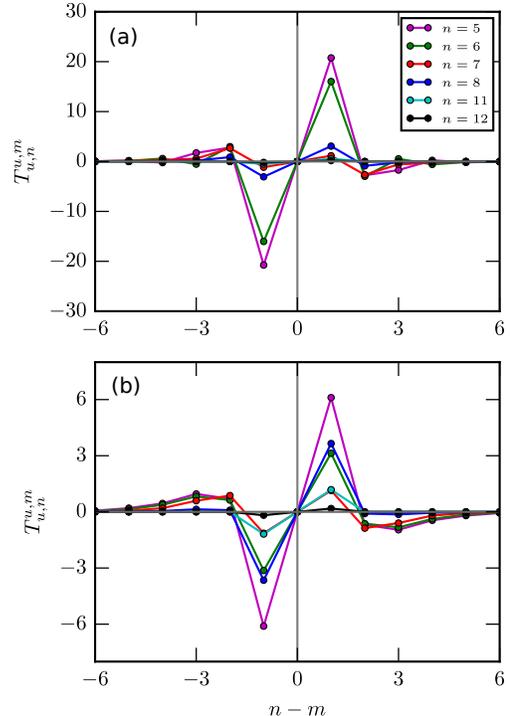


FIG. 4. Plots of shell-to-shell energy transfers $T_{u,n}^{u,m}$ vs. $n - m$ for the 8192^2 grid run in the inertial range shells of $k < k_f$ regime. (a) For a single time frame (b) time-averaged data with 35 frames. The transfers are local and forward for neighbouring shells, but nonlocal and backward for distant shells.

ported by the research grants PLANEX/PHY/2015239 from Indian Space Research Organisation, India, and by the Department of Science and Technology, India (INT/RUS/RSF/P-03) and Russian Science Foundation Russia (RSF-16-41-02012) for the Indo-Russian project, and IITK institute postdoctoral fellowship.

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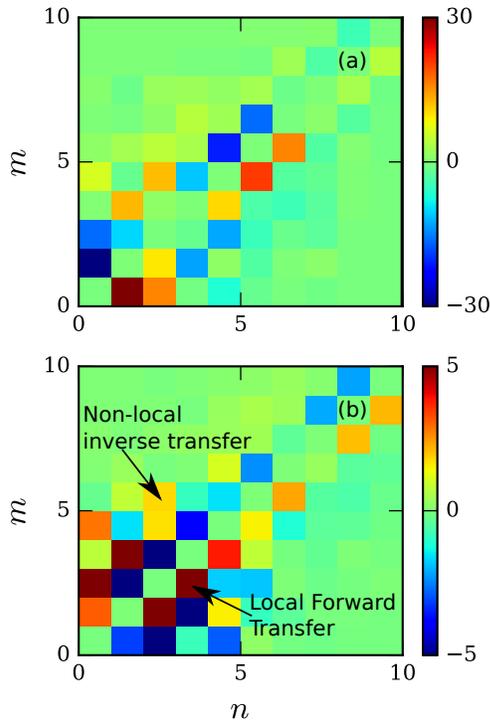


FIG. 5. Density plots of shell-to-shell energy transfers $T_{u,n}^{v,m}$ corresponding to Fig. 4. The x, y axes represent the receiver and giver shells respectively. Here too, the transfers are local and forward for neighbouring shells, but nonlocal and backward for distant shells.

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