

Chapter 5

Hierarchical Financial Structures with Money Cascade



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Abstract In this paper we show similarities between turbulence and financial systems. Motivated by similarities between the two systems, we construct a multiscale model for hierarchical financial structures that exhibits a constant cascade of wealth from large financial entities to small financial entities. According to our model, large and intermediate scale financial institutions have a power law distribution. However, it exhibits Maxwellian distribution at individual scales.

Introduction

A financial system is quite complex due to its multiscale and time-dependent nature. Its complexity is accentuated by features such as saving, banking, corruption, stock-market, natural calamities, etc. Despite such complicated structures, scientists have attempted to model a financial systems using simple ideas. One of the leading questions in this field is how to model the wealth and income distributions of individuals and companies [1]. In this paper we will address this question.

Earlier models of income distribution of individuals are motivated by equilibrium statistical mechanics. In such models, individuals are mapped to particles in a thermodynamic system, and economic activities to scattering among particles. Following this analogy, it is expected that the income distribution follows Maxwellian or Gibbs distribution, similar to the distribution of kinetic energy in a gas container.

The above distribution however holds only for low income groups. Pareto [2] and others showed that the individual income in a large income group exhibits power law distribution. There have been many attempts to model this power law distribution using nonequilibrium nature of the system. See Chakrabarti et al. [1] and references there in.

In a financial system, wealth cascades from large financial entities to smaller ones. This cascade is somewhat similar to the cascade of kinetic energy in a turbu-

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lent system. In addition, a well-developed financial system contains income groups with a wide range of distribution. Also, note that these income groups interact with each other. Motivated by the above similarities between turbulence and finance, we construct a model for a hierarchical financial system which is quite similar to Kolmogorov's model for α turbulent flow.

The structure of the paper is as follows: In section "Equilibrium Model", we describe a generic equilibrium model of a financial system. Section "Multiscale Model of Turbulence" contains a brief description of Kolmogorov's model for turbulence. In section "A Model of Hierarchical Financial Entities" we construct a model for hierarchical financial system; this model is analogous to the Kolmogorov's model of turbulence. We conclude in section "Discussions and Conclusion".

Equilibrium Model

In this section, we describe an equilibrium model of wealth distribution [3]. Before that we discuss thermodynamics of an isolated gas reservoir in which gas molecules interact with each other via collisions. Under thermodynamic approximation, all the molecules in the gas have approximate equal energy. The variation in the energy of the molecules is given by Maxwell or Gibbs distribution [4]:

$$P(E) = \exp(-E/k_B T) \quad (5.1)$$

where $E = mv^2/2$ is the kinetic energy of a molecule of mass m , and $k_B T = \langle mv^2 \rangle / 2$ is the average kinetic energy of all the molecules. Note that this system has a single energy scale $k_B T$. Also, the system is in equilibrium, and it obeys principle of detailed energy balance. As a result, there is no energy transfer from one region to another, both in real and Fourier space.

Now we are ready to describe an equilibrium model of wealth distribution [1, 3]. In the past, several researchers have shown connections between economic systems and equilibrium thermodynamics (e.g., a gas reservoir described above) [3]. The individuals or economic entities are analogous to the gas molecules, and wealth to the kinetic energy of the molecules. Refer to Table 5.1 for a detailed comparison. Using this analogy, researchers deduced that the wealth distribution $P(W)$ in an economy follows Maxwell or Gibbs distribution:

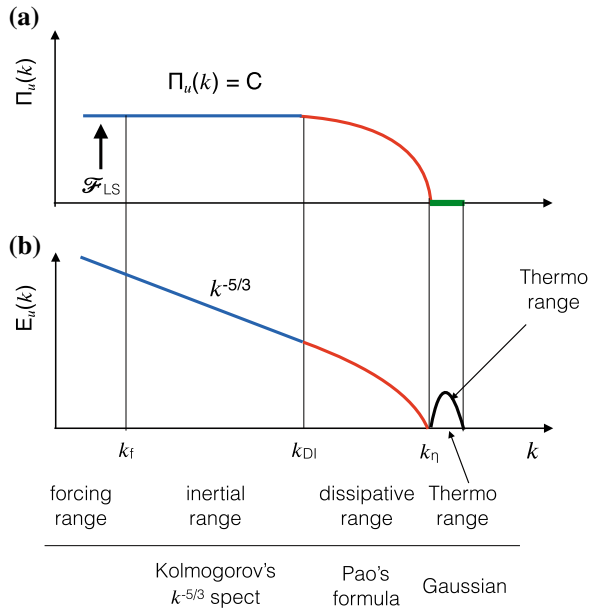
$$P(W) = \exp(-W/\langle W \rangle), \quad (5.2)$$

where $\langle W \rangle$ is the average individual wealth in the economic system. More refined models yield log-normal distribution [1].

Table 5.1 Analogies between an equilibrium economic model and a thermodynamic system

Thermodynamics	Economics
Thermodynamic system	Economy
Gas molecules	Economic entities (individuals)
Individual kinetic energy	Individual wealth
Collisions	Economic interactions
Average kinetic energy	Average wealth

Fig. 5.1 Kolmogorov’s picture of hydrodynamic turbulence. The flow is forced at large scale with an energy injection rate of \mathcal{F}_{LS} . **a** The energy flux is constant in the inertial range, and it decays in the dissipative range. Flux is zero in the thermodynamic range. **b** The energy spectrum exhibits $k^{-5/3}$ spectrum in the inertial range. In the thermodynamic range, the molecules of the fluid exhibit Maxwellian distribution (black curve)



Multiscale Model of Turbulence

Many nonequilibrium systems have properties very different from that of the gas reservoir described above [5]. For example, consider a turbulent fluid stirred at large length scales. The kinetic energy at large scales cascades to intermediate scale, and then to small scales. The kinetic energy flux Π_u is constant in the inertial range, and then it decreases in the dissipation range. Due to the energy cascade, principle of detailed balance is broken in such a system. The energy flux is zero at microscopic scale where we expect thermodynamic principles to hold. This is Kolmogorov’s picture of hydrodynamic turbulence [6–9]. See Fig. 5.1 for an illustration of the energy flux and energy spectrum.

The energy spectrum $E_u(k)$ of a turbulent flow has been derived using dimension analysis. Using

$$[E_u(k)] = [E_u/k] = [L^3/T^2]; \quad [\Pi_u] = [E_u/T] = [L^2/T^3]; \quad [k] = [L]^{-1}, \quad (5.3)$$

we derive the following formula for the kinetic energy spectrum:

$$E_u(k) = K_{\text{Ko}} \epsilon_u^{2/3} k^{-5/3}, \quad (5.4)$$

where K_{Ko} is Kolmogorov's constant, and ϵ_u is the kinetic energy dissipation rate. Pao [10] extended the above formula to the dissipation range, and obtained

$$\Pi_u(k) = \epsilon_u \exp\left(-\frac{3}{2} K_{\text{Ko}} (k/k_d)^{4/3}\right), \quad (5.5)$$

$$E_u(k) = K_{\text{Ko}} \epsilon_u^{2/3} k^{-5/3} \exp\left(-\frac{3}{2} K_{\text{Ko}} (k/k_d)^{4/3}\right), \quad (5.6)$$

where k_d is Kolmogorov's wavenumber [11]. The fluid kinetic energy vanishes beyond the dissipation range, i.e., for $k > k_d$. The above function describes the inertial and dissipative ranges (blue and red curves of Fig. 5.1) quite well. We expect thermodynamic ideas to work beyond this scale. The energy of the molecules would follow Maxwell's or Gibbs' distribution, as shown by the black curve of Fig. 5.1b.

Shell model is a popular model of hydrodynamic turbulence. In one version of the shell model, called GOY shell model of turbulence,

$$\frac{d}{dt} u_n + \nu k_n^2 u_n = -i(a_1 k_n u_{n+1}^* u_{n+2}^* + a_2 k_{n-1} u_{n+1}^* u_{n-1}^* + a_3 k_{n-2} u_{n-1}^* u_{n-2}^*), \quad (5.7)$$

where u_n is a complex number representing the velocity field at length scale $k_n = 2^n$, a_1, a_2, a_3 are constants, and ν is the kinematic viscosity [12]. In the shell model, the kinetic energy follows

$$E(k_n) = \frac{|u_n|^2}{2k_n} \sim k_n^{-5/3}, \quad (5.8)$$

in accordance with Kolmogorov's theory of turbulence. Our finance model has a similar form as the above shell model, as we will describe in the next section.

A Model of Hierarchical Financial Entities

We construct a model for a hierarchical finance system in a similar lines as Kolmogorov's picture for hydrodynamic turbulence. In this model, we assume that the wealth is generated at the largest scale, and then it flows from larger financial structures to smaller structures in a steady manner. We also assume that the financial entities of similar sizes interact with each other. This is similar to the local interactions in turbulence. In addition, in the absence of financial pilferage, we expect the cascade of money from large structures to smaller structures to be a constant. This

Fig. 5.2 In a hierarchical finance model, **a** the flux of money $\Pi_m(k)$, and **b** the wealth distribution of financial entities. The large and medium economic entities have constant money flux and power law wealth distribution of wealth. Small economic entities exhibit exponential distribution, while the thermodynamic range exhibits Maxwellian wealth distribution and zero money flux

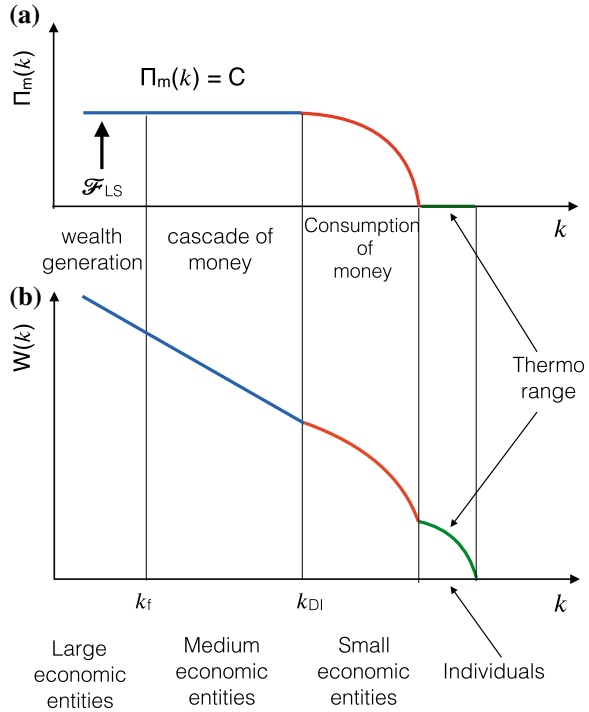


Table 5.2 Analogies between turbulence and hierarchical financial system

Turbulence	Financial system
Fluid structures	Financial entities
Multiscale	Multiscale
kinetic energy of a structure	Wealth of a financial entity
Constant energy flux	Constant money supply
Power law $E_u(k)$ at intermediate scales	Power law for large income entities
Exponential $E_u(k)$ at small scales	Expect similar scaling
Random motion beyond k_d	Gibbs distribution at individual scale

is same as the assumption of constant energy cascade in hydrodynamic turbulence. See Fig. 5.2 for an illustration, and Table 5.2 for a listing of similarities between a turbulent system and a hierarchical financial system.

We place these financial entities in a two-dimensional wavenumber grid.¹ Let us denote the financial asset of a financial entity at the wavenumber \mathbf{k} as $W(\mathbf{k})$. The number of mesh points on a 2D disc of radius k is

¹Dimensionality of a hierarchical financial system is an undetermined parameter. Here we choose $d = 2$ using an observation that these structures reside on the surface of the Earth.

$$n(k) = 2\pi k. \quad (5.9)$$

We solve for the wealth distribution as a function of n . To illustrate, there are fewer financial entities at small k , corresponding to financial giants (e.g. Google and Apple of today). Large number of modes at large k correspond to small units like small companies or individuals.

Motivated by the shell model of turbulence, we construct the following model for the hierarchical financial entities:

$$\frac{dW_k}{dt} = ak^\alpha W_{k-1}W_{k+1} - bk^\beta W_k + Q_{k,1}, \quad (5.10)$$

where a, b, α and β are constants, and W_k is analogous to the shell spectrum in turbulence. Hence,

$$W_k = 2\pi k W(\mathbf{k}). \quad (5.11)$$

In Eq. (5.10), the first term in the RHS represents the interactions among financial entities at scales $k, k-1$ and $k+1$, while the second term represents financial losses at scale k (e.g., recurring expenses, electricity bills). The third term $Q_{k,1}$ represents the wealth generation at the largest scale, $k=1$.

This is a very simple model because it ignores nonlocal interactions, as well as other complex things like loans, savings, banks, generation of wealth at the intermediate and small scales, etc. Further, we assume a steady state in which money flows from larger structures to smaller structures. The wealth is finally consumed at the smallest structures of the system.

First, we focus on the large and intermediate scale where we expect a power law scaling. We also assume that the financial losses at these scales are negligible. Under a steady state,

$$\Pi = \frac{dW_k}{dt} \sim k^\alpha W_k^2, \quad (5.12)$$

where Π is the cascade of money. We invert Eq. (5.12) that yields

$$W_k \sim \Pi^{1/2} k^{-\alpha/2}. \quad (5.13)$$

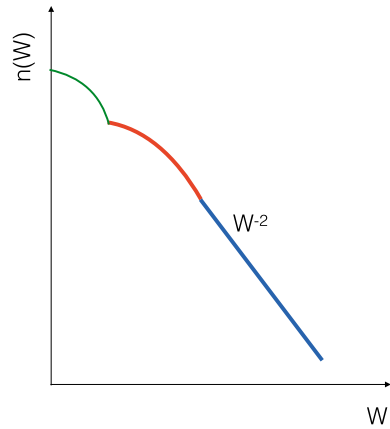
Now using Eqs. (5.9, 5.11), we obtain

$$n(W) \sim \Pi^{\frac{1}{\alpha+2}} W^{-\frac{2}{\alpha+2}}, \quad (5.14)$$

where we write $W(\mathbf{k})$ as W . The above formula yields the number of financial entities $n(W)$ with wealth W . Clearly, $\alpha = -1$ gives

$$n(W) \sim \Pi W^{-2}, \quad (5.15)$$

Fig. 5.3 Plot of wealth distribution $n(W)$ versus W that indicates number of financial entities with wealth W



which is similar to the Pareto’s law for the wealth distribution [1, 2] of the large financial entities. Note however that the exponent depends quite crucially on the choice of α . In Fig. 5.3, we exhibit the inverted form of Fig. 5.9b, or the plot of wealth distribution $n(W)$ versus W .

The wealth cascades down to smaller scales, and it finally gets consumed at the dissipation scales (individual level). It could be in the form of consumption of food and basic needs. Following the popular equilibrium model [3], the wealth distribution at this scale follows Maxwellian or Gibbs distribution. We also expect an income group between the power law regime and the Gibbs distribution. This regime may follow a law similar to that Pao’s model for turbulence, which was discussed in section “Multiscale Model of Turbulence”.

Several cautionary remarks are in order. Our model describes the wealth of financial entities. Pareto’s law however is stated for individual incomes. In free market, a financial entity is essentially owned or controlled by several individuals or a group of individuals. Therefore, it is reasonable to assume that the wealth distribution of financial entities also reflects the wealth or income distribution of individuals. Also, a large financial entity contains smaller entities, thus forming hierarchical structures.

A corollary to the above model is as follows. Let us consider finance distribution in a country. The central government transfers resources to various states who distributes it to lower levels in a hierarchical manner, e.g.,

$$\text{states} \rightarrow \text{districts} \rightarrow \text{villages}. \tag{5.16}$$

Following the same line of arguments as before, we deduce that the financial resources at hierarchical level must be a power law. If there is no corruption, then the money supply at different levels is constant.

The above model is very simple. It ignores many important ingredients such as savings, stocks, banking, pilferage of wealth, nonlocal interactions, etc. This model however has certain novelty. It emphasises on multiscale nature of financial systems,

cascade of money at different scales, and nonequilibrium nature of the financial system.

Discussions and Conclusion

Our finance model, though simple, captures multiscale economic transaction among financial entities and explains coexistence of power law and Maxwellian distribution for the wealth [1]. The model has other predictions as well. Note that the model has a free parameter α . The present multiscale model has many assumptions that need to be studied in detail for applicability in real financial system. For example, we need to include savings, banking, variable energy flux, etc. in a more refined model. In addition, we need to contrast the present model with the existing financial models, some of which are described in [1, 13–15].

A small financial system without hierarchy may exhibit detailed balance and Maxwellian distribution for the wealth distribution. As soon as a financial system becomes sufficiently large and it follows a free-market economy, we expect wealth inequalities to develop based on individual abilities and ambitions. Such a system will exhibit power law distribution. Strong economic regulations may suppress the inequality and make the power law shallower.

We believe that the finance model presented here shed important insights into financial systems. Yet, it is a preliminary model and it needs more work and refinements.

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References

1. Chakrabarti, B.K., Chakraborti, A., Chakravarty, S.R., Chatterjee, A.: *Econophysics of Income and Wealth Distributions*. Cambridge University Press, Cambridge (2013)
2. Pareto, V.: *Cours d'économie politique*. Librairie Droz, Paris (1964)
3. Saha, M., Srivastava, B.N.: *A Treatise on Heat*, 5th edn. Indian Press, Kolkata (1950)
4. Landau, L.D., Lifshitz, E.M.: *Statistical Physics. Course of Theoretical Physics*, 3rd edn. Elsevier, Oxford (1980)
5. Ma, S.-K.: *Statistical Mechanics*. World Scientific, Singapore (1985)
6. Kolmogorov, A.N.: Dissipation of energy in locally isotropic turbulence. *Dokl. Acad. Nauk SSSR* **32**(1), 16–18 (1941)
7. Kolmogorov, A.N.: The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *Dokl. Acad. Nauk SSSR* **30**(4), 301–305 (1941)
8. Lesieur, M.: *Turbulence in Fluids*. Springer, Dordrecht (2008)
9. Verma, M.K.: *Physics of Buoyant Flows: From Instabilities to Turbulence*. World Scientific, Singapore (2018)

10. Pao, Y.-H.: Transfer of turbulent energy and scalar quantities at large wavenumbers. *Phys. Fluids* **11**(6), 1371–1372 (1968)
11. Leslie, D.C.: *Developments in the Theory of Turbulence*. Clarendon Press, Oxford (1973)
12. Ditlevsen, P.D.: *Turbulence and Shell Models*. Cambridge University Press, Cambridge (2010)
13. Chakrabortiand, A., Chakrabarti, B.K.: Statistical mechanics of money: how saving propensity affects its distribution. *Eur. Phys. J. B* **17**, 167–170 (2000)
14. Patriarca, M., Chakraborti, A., Kaski, K.: Statistical model with a standard Γ distribution. *Phys. Rev. E* **70**, 016104 (2004)
15. Chakraborti, A., Patriarca, M.: Variational Principle for the Pareto Power Law. *Phys. Rev. Lett.* **103**, 228701 (2009)