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# The Radial Evolution of the Amplitudes of "Dissipationless" Turbulent Solar Wind Fluctuations

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We examine the evolution with heliocentric distance of the amplitude of fluctuations in the interplanetary magnetic and velocity fields assuming a model of homogeneous, steady state turbulence. Based on observations and recent results, both the Alfvén ratio and the normalized cross helicity are taken to be constant compared to other variations, and the turbulence is assumed to be nearly incompressive. The fluctuation amplitudes are found by taking the heating of the plasma by the fluctuations to be negligible; quasi-steady state turbulence with damping balanced by large-scale energy input will lead to the same conclusions for inertial range fluctuations. While the assumptions of this model contrast sharply with those for purely Alfvénic fluctuations, we find that the radial dependence of the amplitude of the fluctuations for reasonable parameters is very nearly that found from both WKB analysis and recent turbulence modeling. The robustness of this result suggests why some predictions of WKB theory are apparently correct in solar wind conditions where the theory is not expected to be valid.

## INTRODUCTION

Early calculations of the evolution of the amplitudes of interplanetary fluctuations in the magnetic field and plasma variables were based on the paradigm of *Belcher and Davis* [1971] that took the dominant contribution to the fluctuations to be Alfvén waves propagating outward from the Sun. On this assumption, the properly normalized magnetic and velocity fields had the same amplitude and direction everywhere (up to a sign) and were incompressive. Further, the assumption that the spatial scale of the fluctuations was small compared to the large-scale gradients of the Alfvén speed allowed the application of the WKB approximation, which essentially treated the waves as refracting in a medium of slowly varying index of refraction. After a number of such calculations were performed [e.g., *Alazrake and Couturier,* 1971; *Belcher,* 1971] *Hollweg,* [1974] showed that the radial dependence of the fields could be obtained by finding the heating rate of the plasma due to the Alfvén waves and setting it to zero. This showed that the previous calculations were more restrictive than necessary, and in particular that the result was valid for any amplitude and propagation direction for the waves. The existence of an energy conservation argument suggests that the WKB result for the amplitude evolution might be generalized further to take into account a more realistic description of the solar wind; one such generalization is the main point of this paper.

Observational tests of the amplitude evolution show that the WKB prediction works quite well for large-scale fluctuations in the inner heliosphere (spacecraft time scales of days to hours) and for smaller scales (hours to minutes or less) in the outer heliosphere [*Roberts et al.,* 1990 and references therein]. This result is perhaps especially surprising in the outer heliosphere, where the fluctuations are not Alfvénic on average [*Roberts et al.,* 1987*a,b*]. Moreover, it has been argued recently (see for example, *Matthaeus and Goldstein* [1982], *Marsch* [1991], and *Roberts and Goldstein* [1991] for reviews and references) that the evolution of the interplanetary fluctuations is better described by a turbulence rather than a wave model, in accord with *Coleman*'s [1968] original idea. This raises the question of why the WKB model should have any relevance to the evolution of the fluctuations. Some insight into this question was recently obtained by *Zhou and Matthaeus* [1989, 1990]. They derived a transport equation which, in addition to a term involving triple correlations, has a "mixing" term associated with the interaction between large-scale fields and the cross correlation of the small-scale inward and outward traveling waves. They obtained cross correlations using turbulence modeling of the solar wind and thereby obtained an expression for the evolution of the fluctuations in a general varying background. They ignored the triple correlations in their transport equation for the calculation; this assumption depends on either having no nonlinear interactions or having steady state turbulence, where the latter implies that a steady small-scale dissipation of the fluctuations is matched by a steady input from a cascade from the large-scale fields. Under these conditions they showed that the amplitudes of isotropic fluctuations evolve in a manner very similar to that found with WKB theory.

In this paper, we use conservation equations for mass, momentum, and energy to derive an expression for the evolution of the fluctuations, an approach different from *Zhou and Matthaeus* [1989, 1990]. In our calculation, we time average the fluctuating fields at a point; hence we do not have any correlation tensor. The limitation of our calculation as compared to that of Zhou and Matthaeus is that we cannot find the evolution of the spectrum of the fluctuations. On the other hand, our conclusions are independent of any assumption about the wave number spectrum of the turbulence.

We calculate the evolution of the fluctuations not only for

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the equatorial plane but also for the polar region. We find that the fluctuations in the polar region decrease somewhat more slowly than the fluctuations in the equatorial plane, but this effect will be masked by uncertainties in the initial conditions for foreseeable observations.

## Assumptions and Method

The solar wind is known to have a very low level of compressive fluctuations in most regions not near shocks, with the variations in density $\delta\rho \approx 0.1\rho_0$, where $\rho_0$ is the mean background density. A similar fact holds for the magnetic field magnitude such that $\delta B \approx 0.1B_0$ and $\delta(B^2) \approx 0.1(\delta\mathbf{B})^2$, or alternatively the power in the field magnitude is typically one-tenth of that in the component fluctuations [*Goldstein et al.*, 1984]. These observations have recently been interpreted as the signature of the low sonic Mach number of the flow in the solar wind frame [e.g., *Zank and Matthaeus*, 1990 and references therein]; in such cases incompressible MHD is an accurate leading order description of the magnetofluid. This viewpoint is supported by recent MHD simulations [*Roberts et al.*, 1991]. Thus we will take the flow to be locally incompressible and conserving of the magnetic field magnitude even at times when it is not Alfvénic. Consistent with these assumptions, we will also take the fluctuations in the thermal pressure $\delta p$ to be negligible.

The Alfvénic assumption relates the velocity and magnetic fluctuations directly, thus greatly simplifying the analysis of the evolution of the fluctuations. In lieu of this closure, we adopt the observed near constancy of two other quantities involving magnetic and velocity fluctuations, namely the Alfvén ratio $r_A$ between the fluctuating kinetic and magnetic energies, and the normalized cross helicity $\sigma_c$ of the magnetic and velocity fluctuations (the "normalized cross helicity," defined more carefully below). Observations of the Alfvén ratio show that it is very near 1 at 0.3 AU in highly Alfvénic regions and near 0.5 by 1 AU and more generally in less Alfvénic intervals [*Marsch and Tu*, 1990; *Roberts et al.*, 1990]. In Figure 1, the distribution of $r_A$ is shown for 100 days of Voyager 2 data at a distance of about 2 AU from the Sun. The values of $\sigma_c$ also decrease in time, or equivalently with distance as the flow moves outward, but this decrease is much slower than the decrease in such quantities as the amplitude of the magnetic fluctuations, especially in the outer heliosphere. The simplest model of *Zhou and Matthaeus* [1989, 1990] predicts that $\sigma_c$ varies with heliocentric distance $r$ as $r^{-(1/7)}$, and this is generally consistent with the observational results of *Roberts et al.* [1987*a*,*b*]. Thus we take $r_A$ and $\sigma_c$ to be constant for the calculation below.

We assume that the fluctuations have no correlations between the different components; this is strongly supported by the observations of the helicity of the magnetic field [*Matthaeus and Goldstein*, 1982; *Goldstein et al.*, 1991] that show that such correlations are essentially random. In addition, there is a well-established tendency for the minimum variance in the magnetic fluctuations to lie along the mean field direction [*Belcher and Davis*, 1971; *Klein et al.*, 1991, and references therein]. We assume that the ratio between the fluctuations along the mean magnetic field and those perpendicular to it is constant; the constants used in our calculation are 1 (isotropic), 1/3 (close to the actual case), and 0 ("slab" fluctuations).

The one other assumption we need to close the equations is that the time average of some of the products of three

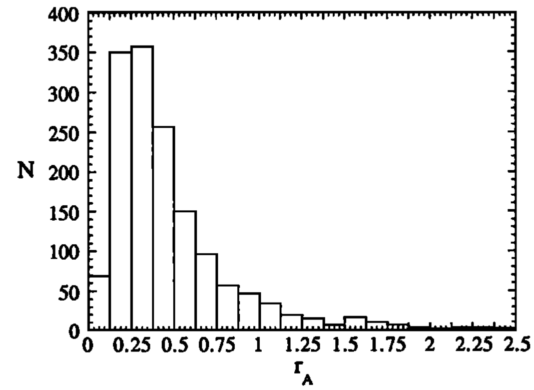


Fig. 1. Histogram showing the distribution of $r_A$ values at the 3-hour scale found using 100 days of Voyager 2 hour-averaged data at a distance of about 2 AU from the Sun. The mean value of $r_A$ is 0.46.

fluctuating vectors at a point (e.g., $\langle v^2\mathbf{b}\rangle$), hereafter referred to as triple products, are negligible compared to other terms. Note that the above mentioned average is not the triple correlation function but rather is the value of the triple correlation when the distance between the correlation points is zero. Hence our assumption is different from *Zhou and Matthaeus'* [1989, 1990] assumption. The assumption that triple correlations are negligible implies that triple products are negligible, but not vice versa. We will show that for "slab" fluctuations the relevant triple products are exactly zero.

As mentioned by *Zhou and Matthaeus* [1989], the triple correlations determine the spectral transfer in the turbulent cascade, and thus neglecting them is to say that the cascade does not matter from the standpoint of the amplitude evolution. This is true in the purely linear limit, but it is also valid for the inertial range of fluctuations if the input to this range from large scales is just enough to match the dissipation that damps fluctuations at the small-scale end of the cascade. Thus if, as may well be the case in the solar wind, the large-scale flows are continually tapped to maintain a quasi-steady spectrum, the cascade will not greatly affect the amplitude of the inertial range fluctuations. In any case, the time scales for spectral transfer are large in the outer heliosphere, and this also would argue for the neglect of triple correlations.

## Analysis

This paper is a generalization of *Hollweg's* [1974] paper. Following his approach, we start with the conservation equations for mass, momentum, and energy and compute time averages of them. Angular brackets will indicate the time average of the quantity enclosed. Throughout the paper, $\mathbf{H}$ denotes the magnetic field in cgs units, $\mathbf{B}$ is the magnetic field in Alfvénic units, $\mathbf{V}$ is the flow velocity of the plasma, $\rho$ is the mass density, and $\mathbf{p}$ is the pressure tensor of the plasma. We separate these quantities into time averaged ($\mathbf{V_0}$, $\mathbf{B_0}$, $\mathbf{H_0}$, etc.) and fluctuating ($\mathbf{v}$, $\mathbf{b}$, $\mathbf{h}$, etc.) parts: $\rho \equiv \rho_0$; $\mathbf{p} \equiv p_0\mathbf{I}$; $\mathbf{B} \equiv \mathbf{H}/\sqrt{(4\pi\rho)}$; $\mathbf{B} \equiv \mathbf{B_0}+\mathbf{b}$; $\mathbf{H} \equiv \mathbf{H_0}+\mathbf{h}$; and $\mathbf{V} \equiv \mathbf{V_0}+\mathbf{v}$. In writing the expressions for the density and the pressure, we have used the assumptions that the incompressible limit gives a good leading order description of a nearly incompressive plasma and that the pressure is isotropic. Since the fluctuating part of all quantities is bounded and stationary, we set $\langle\partial/\partial t\rangle = 0$. We then have

$$\langle\nabla\cdot(\rho\mathbf{V})\rangle = 0 \ , \qquad (1)$$

$$\left\langle \nabla \cdot [\rho \mathbf{V}\mathbf{V} + \mathbf{p} + \frac{1}{2}\rho B^2 \mathbf{I} - \rho \mathbf{B}\mathbf{B}] \right\rangle = \langle \rho \mathbf{g} \rangle, \quad (2)$$

$$\left\langle \nabla \cdot [\frac{1}{2}\rho V^2 \mathbf{V} + \frac{1}{2}\mathrm{Tr}(\mathbf{p})\mathbf{V} + \mathbf{p}\cdot\mathbf{V} + \mathbf{q} + \mathbf{S}] \right\rangle = \langle \rho \mathbf{g}\cdot\mathbf{V} \rangle, \quad (3)$$

where **g** is the local gravitational acceleration, **q** is the heat flux density, and $\mathbf{S} = (c/4\pi)\times(\mathbf{E}\times\mathbf{B})$ is the Poynting flux. Note that it is the equations for the background plasma that we solve, and not the more commonly solved equations for the fluctuations found by subtracting averaged equations from the original equations.

By definition, the mean fluctuation is zero; e.g., $\langle \mathbf{v} \rangle = 0$. Using this, the above equations can be written as

$$\nabla \cdot (\rho_0 \mathbf{V}_0) = 0\ , \quad (4)$$

$$\nabla \cdot \left[ \rho_0 \left( \mathbf{V}_0\mathbf{V}_0 + \langle \mathbf{v}\mathbf{v} \rangle + \frac{1}{2}B_0^2 \mathbf{I} - \mathbf{B}_0\mathbf{B}_0 + \frac{1}{2}\langle b^2 \rangle \mathbf{I} - \langle \mathbf{b}\mathbf{b} \rangle \right) \right] + \nabla p_0 = \rho_0 \mathbf{g}, \quad (5)$$

$$\nabla \cdot \left[ \frac{1}{2}\rho_0 \left\{ \left( V_0^2 + \langle v^2 \rangle \right) \mathbf{V}_0 + 2\left\langle (\mathbf{V}_0\cdot\mathbf{v})\mathbf{v} \right\rangle + \langle v^2 \mathbf{v} \rangle \right\} \right] + \nabla \cdot \left[ \frac{5}{2} p_0 \mathbf{V}_0 + \mathbf{q}_0 + \mathbf{S}_0 \right] = \rho_0 \mathbf{V}_0 \cdot \mathbf{g}\ . \quad (6)$$

We eliminate **g** from (6) by taking the scalar product of (5) with $\mathbf{V}_0$, and inserting the result into (6). This gives

$$\nabla \cdot \left( \frac{3}{2} p_0 \mathbf{V}_0 + \mathbf{q}_0 \right) + p_0 \nabla \cdot \mathbf{V}_0 + \nabla \cdot \left[ \frac{1}{2}\rho_0 \left\{ \mathbf{V}_0 \langle v^2 \rangle + 2\left\langle (\mathbf{V}_0\cdot\mathbf{v})\mathbf{v} \right\rangle + \langle v^2 \mathbf{v} \rangle \right\} + \mathbf{S}_0 \right] =$$
$$\mathbf{V}_0 \cdot \left\{ \nabla \cdot \left[ \rho_0 \left\{ \left( \frac{1}{2} B_0^2 \mathbf{I} - \mathbf{B}_0\mathbf{B}_0 \right) + \frac{1}{2}\langle b^2 \rangle \mathbf{I} + \left( \langle \mathbf{v}\mathbf{v} \rangle - \langle \mathbf{b}\mathbf{b} \rangle \right) \right\} \right] \right\}. \quad (7)$$

We would like to write $\mathbf{S}_0$ in terms of **v** and **b**. Let $\mathbf{E}_0$ denote the average electric field, and **e** denote the fluctuation in the electric field. The average Poynting vector $\mathbf{S}_0$ is given by

$$\mathbf{S}_0 = \frac{c}{4\pi} \left\{ (\mathbf{E}_0 \times \mathbf{H}_0) + \langle \mathbf{e} \times \mathbf{h} \rangle \right\}, \quad (8)$$

where

$$\mathbf{E}_0 = -\frac{1}{c}(\mathbf{V}_0 \times \mathbf{H}_0), \quad (9)$$

and

$$\mathbf{e} = -\frac{1}{c}\left\{ (\mathbf{v}\times\mathbf{H}_0) + (\mathbf{V}_0 \times \mathbf{h}) + (\mathbf{v}\times\mathbf{h}) \right\}. \quad (10)$$

Therefore

$$\mathbf{S}_0 = \rho_0 \left[ \mathbf{V}_0 \left( B_0^2 + \langle b^2 \rangle \right) - \mathbf{B}_0 \left( \mathbf{V}_0 \cdot \mathbf{B}_0 + \langle \mathbf{v}\cdot\mathbf{b} \rangle \right) \right] + \rho_0 \left[ \left\langle \mathbf{v}(\mathbf{B}_0\cdot\mathbf{b}) \right\rangle - \left\langle \mathbf{b}(\mathbf{V}_0\cdot\mathbf{b}) \right\rangle + \langle b^2 \mathbf{v} \rangle - \left\langle \mathbf{b}(\mathbf{v}\cdot\mathbf{b}) \right\rangle \right]. \quad (11)$$

Using $\delta|\mathbf{B}| = 0$, we find that

$$\left( 2\mathbf{B}_0 \cdot \mathbf{b} + b^2 \right) = \mathrm{const} \quad (12)$$

or

$$\left\langle (\mathbf{B}_0 \cdot \mathbf{b})\mathbf{v} \right\rangle = -\frac{1}{2}\left\langle b^2 \mathbf{v} \right\rangle. \quad (13)$$

By inserting the value of $\mathbf{S}_0$ in equation (7) and by using $\mathbf{B}_0\cdot(\nabla\times\mathbf{E}_0) = 0$, which follows from the fact that $\nabla\times\mathbf{E}_0 = 0$ because of time stationarity, we obtain

$$\nabla \cdot \left( \frac{3}{2} p_0 \mathbf{V}_0 + \mathbf{q}_0 \right) + p_0 \nabla \cdot \mathbf{V}_0$$
$$= \mathbf{V}_0 \cdot \left\{ \nabla \cdot \left[ \rho_0 \left\{ \frac{1}{2}\langle b^2 \rangle \mathbf{I} + \langle \mathbf{v}\mathbf{v} \rangle - \langle \mathbf{b}\mathbf{b} \rangle \right\} \right] \right\}$$
$$- \nabla \cdot \left[ \rho_0 \left\{ \frac{1}{2}\langle v^2 \rangle \mathbf{V}_0 + \langle b^2 \rangle \mathbf{V}_0 - \langle \mathbf{v}\cdot\mathbf{b} \rangle \mathbf{B}_0 - \left\langle (\mathbf{V}_0\cdot\mathbf{b})\mathbf{b} \right\rangle \right\} \right]$$
$$- \nabla \cdot \left[ \rho_0 \left\{ \left\langle (\mathbf{V}_0\cdot\mathbf{v})\mathbf{v} \right\rangle + \frac{1}{2}\langle v^2 \mathbf{v} \rangle + \frac{1}{2}\langle b^2 \mathbf{v} \rangle - \left\langle (\mathbf{v}\cdot\mathbf{b})\mathbf{b} \right\rangle \right\} \right]. \quad (14)$$

To write equation (14) in recognizable form, we add $(3/2)\langle \partial p_0/\partial t \rangle$ ($\equiv 0$) to the equation. Now part of equation (14) becomes

$$\frac{\partial}{\partial t}\left( \frac{3}{2} p_0 \right) + \nabla \cdot \left( \frac{3}{2} p_0 \mathbf{V}_0 \right) + p_0 \nabla \cdot \mathbf{V}_0$$
$$= \frac{3}{2}\frac{d}{dt} p_0 - \frac{5}{2}\frac{p_0}{\rho_0}\frac{d}{dt}\rho_0$$
$$= \frac{3}{2}\rho_0^{5/3}\frac{d}{dt}\left(\frac{p_0}{\rho_0^{5/3}}\right). \quad (15)$$

In the second step, we made use of the continuity equation for mass, i.e.,

$$\frac{d}{dt}\rho_0 + \rho_0 \nabla \cdot \mathbf{V}_0 = 0 \quad (16)$$

where $d/dt$ is the convective derivative. However,

$$\frac{3}{2}\rho_0^{5/3}\frac{d}{dt}\left(\frac{p_0}{\rho_0^{5/3}}\right) + \nabla \cdot \mathbf{q}_0 = \mathbf{J}\cdot\mathbf{E}^* \quad (17)$$

where **J** and $\mathbf{E}^*$ are the conduction current density and the electric field respectively in the comoving frame [*Siscoe*, 1983]. From equations (14), (15), and (17), we see that the right hand side of equation (14) is equal to $\mathbf{J}\cdot\mathbf{E}^*$, which is the heating rate due to electromagnetic fields. This heating rate only depends on the fluctuating fields, indicating that the Parker field itself does not heat the plasma. If we assume that there is no dissipation then the right hand side of equation (14) must vanish, giving

$$\nabla \cdot \left[ \rho_0 \left\{ \left( \langle v^2 \rangle + 2\langle b^2 \rangle \right) \mathbf{V}_0 - 2\langle \mathbf{v}\cdot\mathbf{b} \rangle \mathbf{B}_0 - 2\left\langle (\mathbf{V}_0\cdot\mathbf{b})\mathbf{b} \right\rangle \right\} \right]$$
$$+ \nabla \cdot \left[ \rho_0 \left\{ 2\left\langle (\mathbf{V}_0\cdot\mathbf{v})\mathbf{v} \right\rangle + \langle v^2 \mathbf{v} \rangle + \langle b^2 \mathbf{v} \rangle - 2\left\langle (\mathbf{v}\cdot\mathbf{b})\mathbf{b} \right\rangle \right\} \right]$$
$$- \mathbf{V}_0 \cdot \left\{ \nabla \cdot \left[ \rho_0 \left\{ \langle b^2 \rangle \mathbf{I} + 2\left( \langle \mathbf{v}\mathbf{v} \rangle - \langle \mathbf{b}\mathbf{b} \rangle \right) \right\} \right] \right\} = 0. \quad (18)$$

It will be convenient to work in a frame corotating with the sun at angular rate $\Omega_S$. In this frame, $\mathbf{V}_0$ and $\mathbf{B}_0$ are parallel or antiparallel because magnetic field is frozen in the plasma. The additional forces do not modify the equation. The centrifugal acceleration $\Omega_S\times(\Omega_S\times\mathbf{r})$ can be included in $\mathbf{g}$, and the Coriolis force $2\rho(\Omega_S\times\mathbf{V})$ does no work since $(\Omega_S\times\mathbf{V})\cdot\mathbf{V} = 0$.

To simplify equation (18), we make the following assumptions as discussed above:

1. Both $r_A = \langle v^2\rangle/\langle b^2\rangle$ and the normalized cross helicity $\sigma_c = 2\langle \mathbf{v}\cdot\mathbf{b}\rangle/(\langle v^2\rangle+\langle b^2\rangle)$ are constant.

2. The fluctuations have the following properties:

$$\langle b_s^2\rangle = \beta\langle b_n^2\rangle = \gamma\langle b^2\rangle, \tag{19}$$

$$\langle v_s^2\rangle = \beta\langle v_n^2\rangle = \gamma\langle v^2\rangle, \tag{20}$$

$$\langle v_s v_n\rangle = \langle b_s b_n\rangle = 0, \tag{21}$$

where $\beta$ and $\gamma$ are constants; the subscript $s$ denotes the component of the vector along the average flow, and the subscript $n$ denotes either component of the vector perpendicular to the average flow.

3. The triple-product averages $\langle v^2 v_s\rangle$, $\langle b^2 v_s\rangle$, and $\langle(\mathbf{v}\cdot\mathbf{b})v_s\rangle$ are negligible compared to other terms. When the fluctuations along the mean field are zero ("slab turbulence"), these terms are exactly zero.

For isotropic turbulence, $\beta = 1$ and $\gamma = 1/3$ [*Batchelor*, 1953]. We also calculate fluctuations for anisotropic cases in which $\beta = 1/3$, $\gamma = 1/7$ (close to the observed values; see *Klein et al.* [1991]) and $\beta = 0$, $\gamma = 0$ ("slab turbulence"). Using the above assumptions, we can simplify the last term of equation (18) to

$$2(r_A-1)V_0\hat{\mathbf{s}}\cdot\left[\nabla\cdot\left\{\langle h_s^2\rangle\hat{\mathbf{s}}\hat{\mathbf{s}}+\langle h_t^2\rangle\hat{\mathbf{t}}\hat{\mathbf{t}}+\langle h_\theta^2\rangle\hat{\theta}\hat{\theta}\right\}\right] + V_0\frac{d}{ds}\langle h^2\rangle \tag{22}$$

where $\hat{\mathbf{s}}$ is the unit vector along the spiral, $\hat{\mathbf{t}}$ is the unit vector perpendicular to $\hat{\mathbf{s}}$ but in the plane of usual orthogonal unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ (both of these unit vectors are tangent to the cone whose apex angle is $\theta$), and $\hat{\theta}$ is perpendicular to both $\hat{\mathbf{s}}$ and $\hat{\mathbf{t}}$. Note that $V_0$ in equation (22) is the velocity in the corotating frame and is tangential to the spiral. We have used the same symbol $V_0$ for both frames of reference. In the above expression $d/ds$ represents the derivative taken along the direction of flow in the corotating coordinate system. To simplify the above equation, we use vector identity

$$\nabla\cdot(f\mathbf{A}\mathbf{A}) = \mathbf{A}f(\nabla\cdot\mathbf{A})+\mathbf{A}(\mathbf{A}\cdot\nabla)f+f(\mathbf{A}\cdot\nabla)\mathbf{A} \tag{23}$$

where $\mathbf{A}$ is a vector field and $f$ is a scalar field. Using the fact that $\hat{\mathbf{s}}.\hat{\mathbf{t}} = \hat{\mathbf{s}}.\hat{\theta} = 0$, and also $\hat{\mathbf{s}}.\left\{(\hat{\mathbf{s}}.\nabla)\hat{\mathbf{s}}\right\} = 0$ we obtain

$$\text{last term} = V_0\frac{d}{ds}\langle h^2\rangle + 2(r_A-1)V_0\left[\nabla\cdot\left(\langle h_s^2\rangle\hat{\mathbf{s}}\right)+\langle h_t^2\rangle\hat{\mathbf{s}}\cdot\left\{(\hat{\mathbf{t}}\cdot\nabla)\hat{\mathbf{t}}\right\}+\langle h_\theta^2\rangle\hat{\mathbf{s}}\cdot\left\{(\hat{\theta}\cdot\nabla)\hat{\theta}\right\}\right]. \tag{24}$$

After some algebraic manipulation, we obtain

$$\hat{\mathbf{s}}.\left\{(\hat{\theta}.\nabla)\hat{\theta}\right\} = -\cos(\eta)/r \tag{25}$$

and

$$\hat{\mathbf{s}}\cdot\left\{(\hat{\mathbf{t}}\cdot\nabla)\hat{\mathbf{t}}\right\} = \sin(\eta)\frac{\partial\eta}{\partial r}-\frac{\cos(\eta)}{r} \tag{26}$$

where $\eta$ is the angle between $\hat{\mathbf{s}}$ and $\hat{\mathbf{r}}$. We use the formula $\tan(\eta) = r\Omega_S\sin(\theta)/V_{SW}$ where $V_{SW}$ is the speed of the solar wind. Using the above, equation (26) becomes

$$\hat{\mathbf{s}}.\left\{(\hat{\mathbf{t}}.\nabla)\hat{\mathbf{t}}\right\} = -\frac{\cos^3(\eta)}{r} \tag{27}$$

Using (24), (25), and (27), we find that

$$\text{last term} = V_0\frac{d}{ds}\langle h^2\rangle + 2(r_A-1)V_0\left[\frac{1}{A}\frac{d}{ds}\left(A\langle h_s^2\rangle\right)-\langle h_t^2\rangle\frac{\cos^3(\eta)}{r}-\langle h_\theta^2\rangle\frac{\cos(\eta)}{r}\right] \tag{28}$$

where $A$ is the cross-sectional area of the flow tube along which the derivative is taken. Now substituting (28) into equation (18), and also using equation (19) which implies that $\langle h_t^2\rangle = \langle h_\theta^2\rangle = \langle h_n^2\rangle$, we obtain

$$\frac{1}{A}\frac{d}{ds}\left[A\langle h^2\rangle\left\{(2+r_A)V_0-(1+r_A)\sigma_c B_0\right\}\right] - V_0\frac{d}{ds}\langle h^2\rangle + 2(r_A-1)V_0\langle h_n^2\rangle\frac{\cos(\eta)}{r}\left(1+\cos^2(\eta)\right) = 0. \tag{29}$$

Using $\nabla\cdot(\rho_0\mathbf{V}_0) = \nabla\cdot\mathbf{H}_0 = 0$, we obtain

$$\rho_0 V_0 A = \text{const} \tag{30}$$

and

$$B_0 A\sqrt{\rho_0} = \text{const}. \tag{31}$$

Therefore

$$\frac{AB_0}{\sqrt{AV_0}} = \text{const}. \tag{32}$$

After simplification of equation (29) using equation (32), we obtain

$$(1+r_A)\left(1-\frac{\sigma_c}{M}\right)\frac{d}{ds}\left(\ln\langle h^2\rangle\right) + \left\{(2+r_A)-\frac{(1+r_A)}{2}\frac{\sigma_c B_0}{V_0}\right\}\frac{d}{ds}\left(\ln(AV_0)\right) + 2(r_A-1)\delta\frac{\cos(\eta)}{r}\left(1+\cos^2(\eta)\right) = 0. \tag{33}$$

Also we take $\langle h_n^2\rangle = \delta\langle h^2\rangle$, where $\delta = (1-\gamma)/2$ is a constant (from equations (19) and (20)). Since $B_0/V_0$ is roughly 1/10 near and beyond 1 AU, and further $\sigma_c$ tends to be much less than 1 in the outer heliosphere, we can ignore $\sigma_c B_0/V_0$. Since there is no velocity along $\hat{\theta}$, the spiral is contained in a particular cone. Hence for all the spirals in a given cone, the half angle $\theta$ can be treated as a constant parameter, and $r$ and $s$ have a one-to-one correspondence, which enables us to change the variables from $r$ to $s$. Using $dr/ds = \cos(\eta)$, we obtain

$$\frac{d}{dr}(\ln\langle h^2\rangle)+\frac{(2+r_A)}{(1+r_A)}\frac{d}{dr}(\ln(AV_0))+2\delta\frac{(r_A-1)}{(1+r_A)}\frac{d}{dr}\ln(r)$$
$$-\delta\frac{(r_A-1)}{(1+r_A)}\frac{d}{dr}\ln\left\{1+\left(\frac{V_{sw}}{r\Omega_s\, sin\theta}\right)^2\right\}=0. \quad (34)$$

Using $AV_0 \propto \rho_0^{-1} \propto r^2$ and choosing $r_0$ as a reference point, we finally obtain $\langle h^2(r,\theta)\rangle$ in terms of $\langle h^2(r_0,\theta)\rangle$:

$$\langle h^2(r,\theta)\rangle=\langle h^2(r_0,\theta)\rangle$$
$$\times\left(\frac{r}{r_0}\right)^{-\left[2\frac{(2+r_A)}{(1+r_A)}-2\delta\frac{(1-r_A)}{(1+r_A)}\right]}\left[\frac{1+\left(\frac{Vsw}{r\Omega_s \sin\theta}\right)^2}{1+\left(\frac{Vsw}{r_0\Omega_s \sin\theta}\right)^2}\right]^{-\delta\frac{(1-r_A)}{(1+r_A)}}. \quad (35)$$

For Alfvénic or any other equipartitioned fluctuations, $r_A$ = 1. This leads to the standard result

$$\langle h^2\rangle \propto r^{-3}. \quad (36)$$

Note, however, that this holds no matter what assumption is made about the isotropy of the fluctuations.

In the equatorial plane ($\theta = \pi/2$), in the limit $r >> 1$ AU, we obtain

$$\langle h^2(r,\theta)\rangle=\langle h^2(r_0,\theta)\rangle r^{-\left[2\frac{(2+r_A)}{(1+r_A)}-2\delta\frac{(1-r_A)}{(1+r_A)}\right]}. \quad (37)$$

Suppose we take a typical Alfvén ratio of 0.5 for the solar wind (see Figure 1). Assuming isotropic turbulence ($\delta = 1/3$), we obtain

$$\langle h^2\rangle \propto r^{-\frac{28}{9}}. \quad (38)$$

If we assume that the fluctuations along the mean field direction are absent ($\delta = 1/2$), we obtain

$$\langle h^2\rangle \propto r^{-3}. \quad (39)$$

Finally, if fluctuations along the field are assumed to be one-third those in the transverse direction ($\delta = 3/7$), a case close to that observed, we obtain

$$\langle h^2\rangle \propto r^{-\frac{64}{21}}. \quad (40)$$

None of the above results differs from the standard WKB result enough to be observationally distinguishable from it, and all of them are consistent with results from Voyager data [see *Roberts et al.*, 1990].

The evolution in the polar region (i.e., $r\,\Omega_S \sin\theta \ll V_{SW}$) is given by

$$\langle h^2(r,\theta)\rangle=\langle h^2(r_0,\theta)\rangle r^{-\left[2\frac{(2+r_A)}{(1+r_A)}-4\delta\frac{(1-r_A)}{(1+r_A)}\right]}. \quad (41)$$

We see that the fluctuations in the polar region typically decrease more slowly than that in the equatorial plane. If we take the fluctuations to have an amplitude independent of $\theta$ at 0.1 AU, then the difference between the amplitudes at the ecliptic and over the poles will only amount to about 10% at 2.5 AU, and thus this effect will not be observable by Ulysses. By 40 AU the difference is about a factor of 3 in power level, and thus there may be some consequences of this effect for the outer heliosphere. However, the uncertainties in the level of the fluctuations at the source and in the differences between the turbulent evolution in the two cases are sufficiently great that we cannot yet arrive at any clear conclusions about the angular dependence of the fluctuations.

Figures 2 and 3 show plots of the power law index $\alpha$ versus $r_A$ in the formula $\langle h^2\rangle \propto r^{-\alpha}$ for equatorial and polar region respectively. The figures shows the remarkably small range of values for this index as $\delta$ and $r_A$ vary over the observed range. The result from the *Zhou and Matthaeus* [1989, 1990] turbulence modeling approach is also shown for comparison as $\alpha_{zm}$; note the qualitative similarity of our results to theirs.

## CONCLUSION

We have found that the evolution of the amplitudes of interplanetary fluctuations is very close to the WKB result under the more realistic assumption of nearly incompressible, steady state turbulence. The case in which the fluctuations have nearly the observed anisotropy is indistinguishable from the

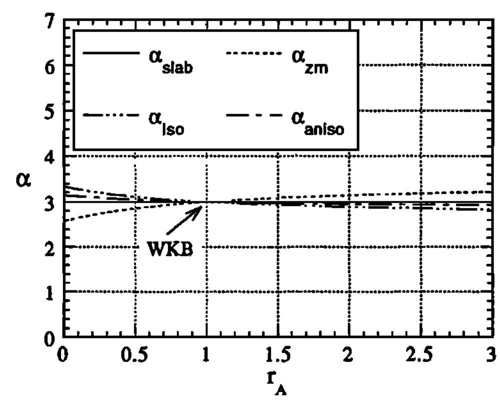


Fig. 2. For the equatorial plane, plot of the power law index, $\alpha = \{2(2+r_A)-2\delta(1-r_A)\}/(1+r_A)$ in the formula $\langle h^2\rangle \propto r^{-\alpha}$, as a function of $r_A$. The lines are for the "slab" ($\delta = 1/2$), isotropic ($\delta = 1/3$), and anisotropic ($\delta = 3/7$) fluctuations. Note that for the "slab" turbulence $\alpha$ is exactly 3 for all $r_A$. Zhou and Matthaeus' [1989, 1990] result is given for comparison by the $\alpha_{zm}$ curve, and all the curves intersect at the point representing the WKB result.

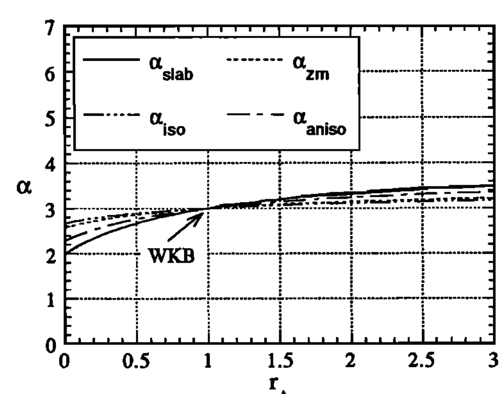


Fig. 3. For the polar region, plot of the power law index $\alpha = \{2(2+r_A)-2\delta(1-r_A)\}/(1+r_A)$ in the formula $\langle h^2\rangle \propto r^{-\alpha}$, as a function of $r_A$.

WKB case, no matter what the value of $r_A$. The most important assumption involved is that the fluctuations do not heat the plasma, and thus what we calculated was the adiabatic behavior. The result is also largely independent of the assumption made about the isotropy of the fluctuations. Thus the observed amplitude evolution in the outer heliosphere can be explained by nearly incompressible turbulent fluctuations under plausible assumptions. Unlike previous calculations, this one makes no assumption about the distribution of the turbulence in wave vector space, and in particular does not assume a Kolmogoroff power law spectrum or isotropy. (Conversely, it cannot determine the behavior in this space.) This work could be extended to regions where Joule heating occurs in the plasma by modeling the Joule heating term. This may help us to understand better the evolution of the fluctuations in solar wind fields in the inner heliosphere.

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