

## Large-eddy simulations of fluid and magnetohydrodynamic turbulence using renormalized parameters

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**Abstract.** In this paper a procedure for large-eddy simulation (LES) has been devised for fluid and magnetohydrodynamic turbulence in Fourier space using the renormalized parameters. The parameters calculated using field theory have been taken from recent papers by Verma [1,2]. We have carried out LES on  $64^3$  grid. These results match quite well with direct numerical simulations of  $128^3$ . We show that proper choice of parameter is necessary in LES.

**Keywords.** Turbulence; large-eddy simulations.

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Turbulence is one of the most difficult and unsolved problems of classical physics. To probe the complex dynamics of turbulence, one often resorts to computer experiments, known as direct numerical simulations (DNS). Since multiple scales are involved in turbulence, DNS of turbulence is a very expensive task in terms of both computer time and memory, even in modern computers. For example, a pseudo-spectral simulation by Gotoh [3] on  $1024^3$  grid using vector parallel Fujitsu VPP5000/56 with 32 processors took 500 h of computer time, and required 8 Gigabytes of memory per processor. To reduce the required computer time and memory space, an ingenious technique called large-eddy simulation (LES) has been developed (see review article by Métais [4] and references therein).

Basic idea of LES is to resolve only the large scales of turbulent flow. The effect of smaller scale interactions are modeled appropriately using the existing theories. In turbulence, Fourier modes of different scales interact with each other. Kolmogorov provided an important model of turbulence in which the interactions effectively yield a constant energy flux from large scales to intermediate scales, and then to small scales. When we observe Fourier modes up to certain length scale  $l$  in the intermediate range, the modes with scales less than  $l$  act as a sink of energy. According to Kolmogorov's theory, the amount of sink should be equal to the energy

flux. In LES, the large scales up to  $l$  are resolved by using eddy viscosity at cut-off scale  $l$ , where energy is drained. Analysis of turbulence using renormalization groups (RG) shows that the above modeling is possible. LES uses this idea to analyse large-scale dynamics of turbulence.

Renormalization group (RG) is a popular tool used by physicists to solve problems with multiple scales. Since turbulence involves multiple scales, RG has been applied successfully to turbulence [5–7]. In Wilson’s Fourier-space RG scheme, Fourier space is divided into many shells. The non-linear interactions among various shells are computed using first-order perturbation theory, that yields an effective viscosity, called renormalized or eddy viscosity, at any scale. The renormalized viscosity is found to be wavenumber ( $k$ ) dependent. McComb and Watt [8] computed the renormalized viscosity using ‘self-consistent’ RG procedure. When the cut-off wavenumber  $k_C$  is in the inertial range, the renormalized viscosity is given by

$$\nu_r(k_C) = (K)^{1/2} \Pi^{1/3} k_C^{-4/3} \nu^*, \quad (1)$$

where  $\Pi$  is the energy flux,  $K$  the Kolmogorov’s constant, and  $\nu^*$  a parameter. McComb and Watt [8] found  $\nu^* \approx 0.50$  and  $K \approx 1.62$ . Verma [1] also computed the above quantities using a refined technique and found  $\nu^* \approx 0.38$  and  $K \approx 1.6$ .

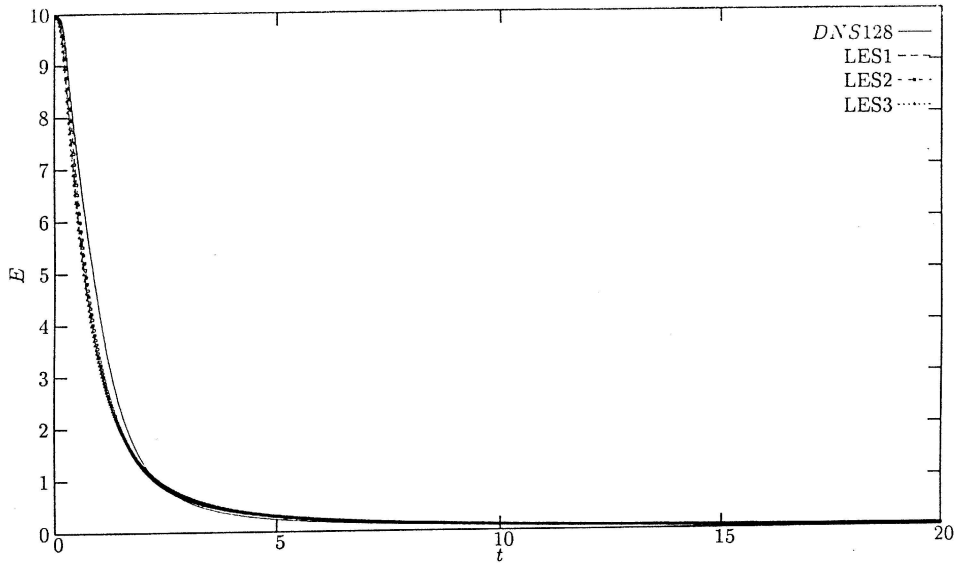
Zhou and Vahala [9,10] developed an alternative recursive renormalization group theory for turbulence modeling. In their calculation they find backscatter of energy from small scales to large scales, and a cusp in renormalized viscosity near  $k_C$ . These features are attributed to triple correlations, which have not been accounted for in McComb and Watt’s calculations. In fact, cusp in effective viscosity was first proposed by Kriachnan [11] while studying energy transfers in turbulence. Recently, Schilling and Zhou [12] have addressed the above problem using eddy-damped quasi-normal Markovian (EDQNM) closure model. In the current paper we neglect backscatter.

McComb [7] had proposed that the renormalized viscosity  $\nu(k_C)$  could be used as effective viscosity for LES. However, this calculation had not been done till date. Earlier, the spectral eddy viscosity  $\nu_t(k|k_C)$  has been used for LES in EDQNM formalism (see [13]). In this scheme,

$$\nu_t(k|k_C) = 0.441 K^{-3/2} \left[ \frac{E(k_C)}{k_C} \right]^{1/2} f(k/k_C), \quad (2)$$

where  $f(x)$  is a non-dimensional function which tends to 1 as  $x$  approaches 0. Comparing eqs (1) and (2), we find that their dependence on Kolmogorov’s constant is different. In eq. (2), if we assume that  $E(k_C)$  follows Kolmogorov’s spectrum and  $K = 1.6$ , then the constant multiplying  $\Pi^{1/3} k_C^{-4/3}$  is 0.27. In contrast, in eq. (1) the same quantity is  $\sqrt{K} \nu^* \approx 0.48$ . As will be shown in the later part of the paper, the choice of constant is quite crucial in LES. We find that  $\nu_r(k_C) = 0.38$  of eq. (1) yields better numerical results compared to  $\nu_t(k|k_C) = 0.48$  of eq. (2). We believe that the calculation of renormalized viscosity is theoretically more sound than the calculation of spectral eddy viscosity using EDQNM approximation. Therefore, the former is more appropriate for LES than the latter.

A major problem in using renormalized parameters in LES is that most of the RG procedure assume homogeneity and isotropy, which is not applicable in practical



**Figure 1.** Temporal evolution of energy in decaying fluid turbulence using DNS and LES. The figure contains energy ( $E$ ) vs. time ( $t$ ) plots for DNS128 (solid line), and three LES runs using  $\nu^*$  equal to 0.38 (LES1), 0.25 (LES2) and 0.48 (LES3). The evolution in LES for all the three  $\nu^*$  is quite close to DNS128, but not close to DNS64.

problems. We need to use the results from some of the anisotropic RG calculations [14], and apply them to LES.

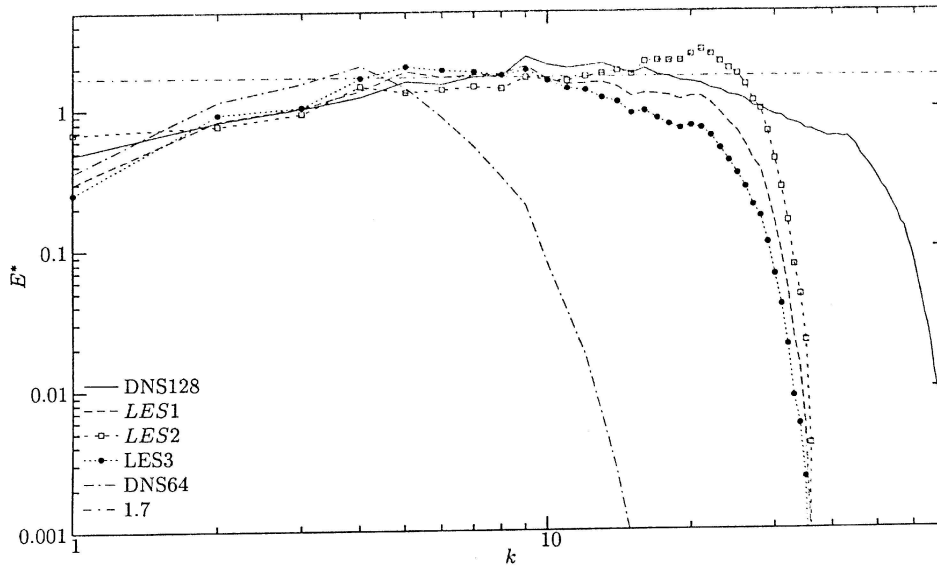
In this paper we perform LES of fluid turbulence using renormalized viscosity. Then we apply the same procedure to magnetohydrodynamic (MHD) turbulence also, except that we need two renormalized parameters: renormalized viscosity and renormalized resistivity. The required parameters for MHD have been recently calculated by Verma [1,2,15]. The LES calculations have been performed on  $64^3$  grid, and they have been compared with DNS results of  $64^3$  and  $128^3$ . As described below, the inertial range in LES is found to be either equal to or larger than that in DNS. Hence our LES model is working very well.

We solve Navier–Stokes equation in Fourier space [16]:

$$\frac{\partial u_i(\mathbf{k})}{\partial t} = -\nu_r(k_C)k^2 u_i(\mathbf{k}) - \text{FT} \left( u_j \frac{\partial u_i}{\partial x_j} \right) - ik_i p(\mathbf{k}), \quad (3)$$

where FT stands for Fourier transform. We take  $\nu^*$  to be equal to 0.38.

We adopt pseudo-spectral method on grid size  $64^3$  with  $dt = 10^{-4}$ . We apply Adam–Bashforth scheme to integrate the non-linear terms, and Crank–Nicholson’s scheme for the viscous term. For LES calculations, we apply 2/3 rule to eliminate the aliasing errors [16]. We use fast Fourier transform developed by Frigo and Johnson [17] for our calculations. Our initial condition is taken to be 10 units energy spread out in wavenumber shells from 2 to 13 with an exponentially decreasing distribution. The modes in a shell have equal energy but random phases, and



**Figure 2.** Normalized energy spectrum  $E^* = E(k)k^{5/3}\Pi^{-2/3}$  vs.  $k$  plots for fully developed fluid turbulence using DNS and LES. The spectrum was calculated at 50 time units for DNS128, DNS64, and three LES runs using  $\nu^*$  equal to 0.38 (LES1), 0.25 (LES2) and 0.48 (LES3). We get the best inertial range for  $\nu^* = 0.38$ . The Kolmogorov's constant is found to be  $1.7 \pm 0.1$ . Note that DNS64 run has hardly any inertial range.

satisfy divergenceless condition. The most important ingredient in our simulation is renormalized viscosity, which is computed using eq. (1) with  $k_C = 32$ . Since forcing is not applied to Navier–Stokes equation, ours is a decaying simulation. Since  $\Pi$  changes with time, it is computed every 0.01 dimensionless time unit. We use dissipation rate for  $\Pi$ . We carry out our simulation up to 20 time units. Our LES simulation takes approximately 3 h per time unit on Athlon 1.7 GHz processor.

In figure 1 we show the energy evolution as a function of time for  $\nu^* = 0.25, 0.38, 0.48$ . The  $E$  vs.  $t$  plot for all three  $\nu^*$  are overlapping. The LES results are also compared with the standard pseudo-spectral DNS results performed on  $128^3$  (DNS128) with identical initial condition and  $\nu_0 = 4 \times 10^{-4}$ . In DNS we apply additional hyperviscous term  $\nu_0/k_{\text{eq}}^2 k^4 \mathbf{u}(\mathbf{k})$  with  $k_{\text{eq}} = 7$  to overcome aliasing errors. Clearly the energy evolution for LES matches quite well with DNS128. Hence, our LES on  $64^3$  is able to mimic DNS of  $128^3$ .

In figure 2, we plot  $E(k)k^{5/3}\Pi^{-2/3}$  vs.  $k$  for DNS as well as LES. Again, the normalized spectrum of LES matches quite well with DNS128 at small and intermediate wavenumbers. Note that  $64^3$  LES has much larger inertial range compared to  $64^3$  DNS, where it is almost absent. We find that the wavenumber range of inertial wavenumbers (constant with  $k$ ) is maximum for LES with  $\nu^* = 0.38$ . The energy spectrum for  $\nu^* = 0.25$  has a hump for large wavenumbers (underdamped case), implying that actual  $\nu^*$  value is higher than 0.25. The spectrum for  $\nu^* = 0.48$  shows overdamped character [18]. We have done DNS128 for some more

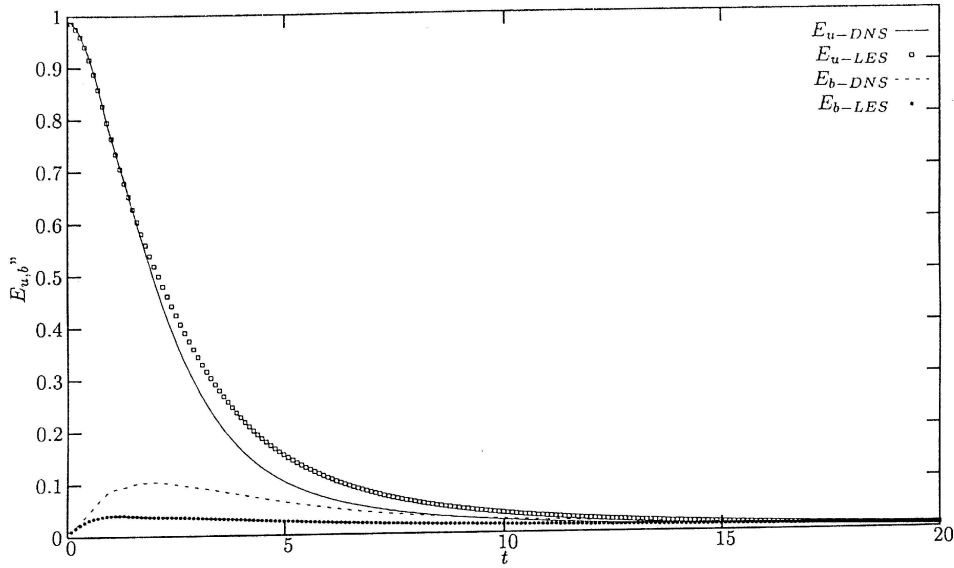
**Table 1.** The values of renormalized parameters for viscosity ( $\nu^*$ ) and resistivity  $\eta^*$  in MHD turbulence at various values of Alfvén ratio  $r_A$  and zero cross helicity. We also list the Kolmogorov's constant  $K^u$  for MHD turbulence.

$r_A$	$\nu^*$	$\eta^*$	$K^u$
0.3	7.20	0.20	0.50
0.4	3.15	0.38	0.53
0.5	2.08	0.50	0.55
0.6	1.64	0.57	0.59
0.7	1.38	0.61	0.63
0.8	1.21	0.64	0.67
0.9	1.09	0.67	0.71
1.0	1.00	0.69	0.75
2.0	0.65	0.77	1.01
3.0	0.54	0.79	1.15
4.0	0.49	0.81	1.23
5.0	0.47	0.82	1.28

parameters. The trend appears to show that  $\nu^* \approx 0.38$  is the most appropriate choice for LES. Fortunately, we obtain the above value using renormalization group calculation [2]. It is interesting to note from figure 1 that the temporal evolution of energy does not clearly tell us which  $\nu^*$  is the most appropriate for LES. Hence we should be careful in concluding the appropriateness of  $\nu^*$  using energy evolution. The energy spectrum has more information, and can provide us clues on the correct choice of  $\nu^*$ .

From figure 2 we obtain the numerical value of  $K$  to be  $1.7 \pm 0.1$ ; this value is close to the theoretically calculated value 1.6 by many authors. Hence, the renormalized viscosity predicted by Verma [1] appears to be consistent and provides us a very good scheme for LES.

For LES of magnetohydrodynamic (MHD) turbulence, Agullo *et al* [19] and Müller and Carati [20,21] applied dynamic gradient-diffusion subgrid model. The forms of eddy-viscosity and eddy-resistivity are derived using dimensional arguments, but the constants are calculated using dynamical LES procedure. Their results match very well with the DNS counterpart. In one of their main models, turbulent viscosity  $\nu_t \approx \bar{l}^{4/3}(\epsilon^K)^{1/3}$  and turbulent resistivity  $\eta_t \approx \bar{l}^{4/3}(\epsilon^M)^{1/3}$ , where  $\bar{l}$  is the resolvable length scale on the LES grid, and  $\epsilon^K$  and  $\epsilon^M$  are kinetic and magnetic energy dissipation applied by the subgrid scale respectively. Zhou *et al* [22] have studied subgrid scale and backscatter model for MHD turbulence using EDQNM closure scheme. Verma [1,2] has also calculated the above parameters using renormalization group procedure. Simple calculations show that turbulent dissipative parameters of Verma differ from those of Agullo *et al* [19] and Müller and Carati [20,21], as well as from those of Zhou *et al* [22]. In the following discussions we will compare the LES results from the above three approaches.



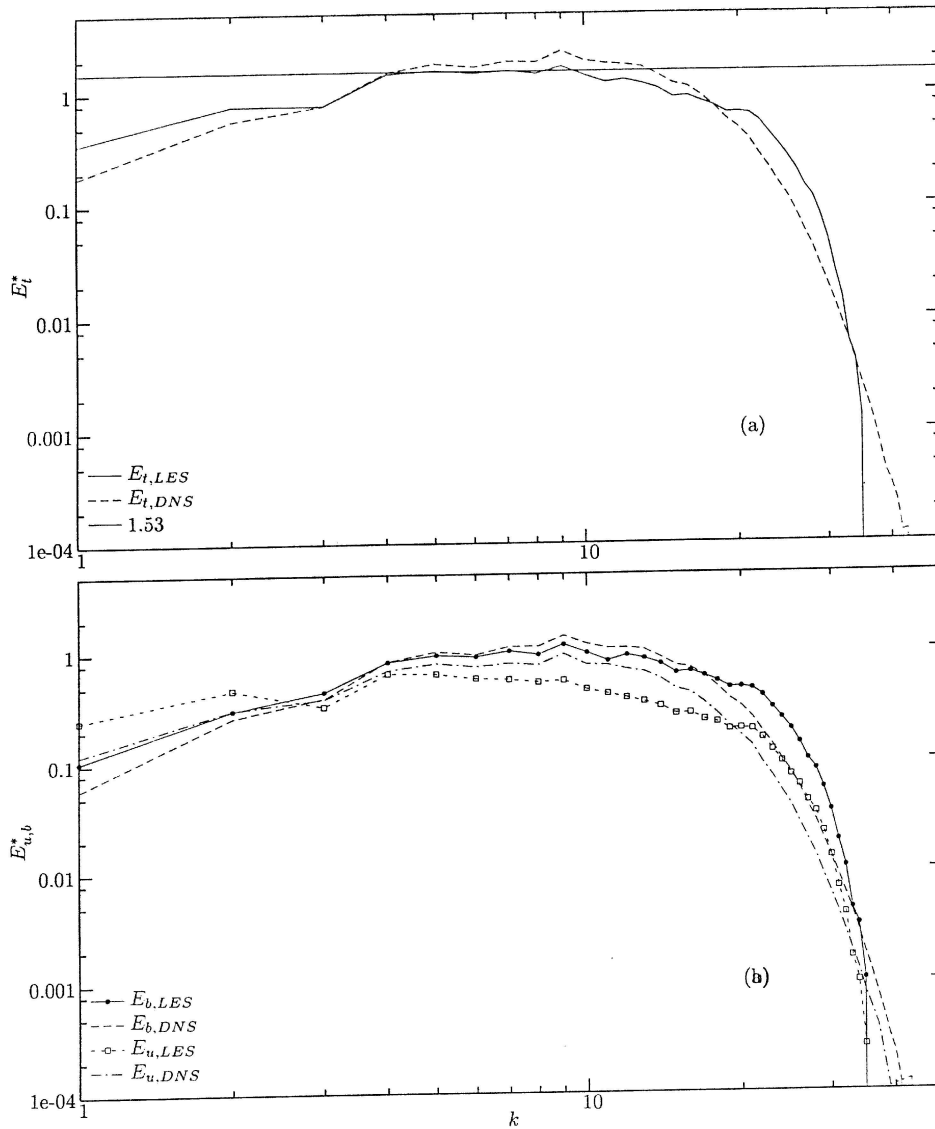
**Figure 3.** Temporal evolution of total kinetic and magnetic energy in decaying MHD turbulence using DNS and LES. The kinetic energy matches quite well in both the schemes, but magnetic energy evolves somewhat differently.

For MHD turbulence we apply the same LES method as described for fluid turbulence using renormalized parameters. The pseudo-spectral method to solve MHD equations is very similar to that of fluid turbulence. We also confine ourselves to zero cross helicity, i.e.,  $\mathbf{u} \cdot \mathbf{b} = 0$ , and zero mean magnetic field. The difference of LES and DNS is in the values of viscosity and resistivity. In DNS we take  $\nu_0 = 0.0004$  and  $\eta_0 = 0.0004$  with hyperviscosity and hyperresistivity parameters  $k_{\text{eq}} = 7$ . However, in LES we take  $\nu(k_C) = \nu_r(k_C)$ , and  $\eta(k_C) = \eta_r(k_C)$ , where  $k_C$  is the cut-off wavelength. The renormalized viscosity  $\nu_r(k_C)$ , and renormalized resistivity  $\eta_r(k_C)$  are taken from Verma [1,2] as

$$\nu_r(k_C) = (K^u)^{1/2} \Pi^{1/3} k_C^{-4/3} \nu^* \quad (4)$$

$$\eta_r(k_C) = (K^u)^{1/2} \Pi^{1/3} k_C^{-4/3} \eta^* \quad (5)$$

Here  $K^u$  is the Kolmogorov's constant for MHD,  $\Pi$  is the total energy flux, and  $\nu^*, \eta^*$  are renormalized parameters. The parameters  $\nu^*, \eta^*$ , and  $K^u$  depend on the Alfvén ratio  $r_A$ , which is the ratio of kinetic and magnetic energy. In our decaying MHD turbulence simulation, we start with unit total energy and  $r_A = 100$ . The ratio of magnetic to kinetic energy grows as a function of time, as expected. Therefore, we need to compute the renormalized parameters for various values of  $r_A$ . The parameters have been calculated using the procedure described in Verma [2], and they are shown in table 1. We use the appropriate  $\nu^*$  and  $\eta^*$  given in the table for our simulations. The energy cascade rates are computed using the flux formula derived in Dar *et al* [23]. It involves computation of non-linear terms using fast Fourier transforms [17]. We take  $\nu_r(k_C)$  and  $\eta_r(k_C)$  from eqs (4) and (5). The



**Figure 4.** Plots of normalized spectra  $E^*(k) = E(k)k^{5/3}\Pi^{-2/3}$  with wavenumber  $k$  for MHD turbulence. Figure 4a is the plot for total energy spectrum and figure 4b is the plot for magnetic and kinetic energy spectra.

energy flux  $\Pi$  changes with time; we compute  $\Pi$  dynamically every 0.01 time unit. We carried out LES for MHD up to 25 non-dimensional time units, and it took approximately 55 h.

The evolution of kinetic and magnetic energies are shown in figure 3 as a function of time. The evolution of kinetic energy using LES is quite close to that using DNS. However, the evolution of magnetic energy does not match very well. Compara-

tively, LES of Agullo *et al* [19] and Müller and Carati [20,21] yield a better fit to the temporal evolution of energy. Figure 4 shows the energy spectra of total energy (figure 4a), and kinetic and magnetic energies (figure 4b) for  $r_A = 1$  at 10 time units of DNS and 17 time units of LES. We find that the energy spectra calculated in LES matches quite well with that in DNS. Note however that the agreement for the fluid energy is not very good. The Kolmogorov's constant as indicated by the straight line in figure 4a is found to be  $1.51 \pm 0.05$ , which is close to the theoretical value calculated in [1,2]. We conclude that the LES based on renormalized parameters of Verma [1,2] is quite good. Our numerical results are comparable with results of Agullo *et al* [19] and Müller and Carati [20,21]. However, we believe that our parameters, which are based on field-theoretic calculations, are on a somewhat stronger footing as compared to those used in earlier LES methods.

To conclude, we have devised a LES procedure for fluid and MHD turbulence in Fourier space using the renormalized parameters. We take renormalized parameters from Verma [1,2] and carry out LES for  $64^3$  grid. When LES results are compared with DNS of size  $128^3$  with the same initial conditions, we find that our LES results on energy evolution and spectra match quite well with the DNS results, except for the temporal evolution of magnetic energy. The inertial range of LES is much larger compared to DNS of the same size. Our results show that substitution of renormalized parameters for eddy viscosity in LES yield excellent results. When the effects of triple correlations on renormalized parameters will be included, we would obtain even better fits. Hence, we demonstrate the usefulness of renormalized parameters in LES calculations. Ours is only a preliminary calculation in this direction. We need to explore further to apply these ideas to practical problems, for which we need renormalized parameters in anisotropic and more complex conditions.

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### References

- [1] M K Verma, *Phys. Rev.* **E64**, 26305 (2001)
- [2] M K Verma, *Phys. Plasma* **8**, 3945 (2001)
- [3] T Gotoh, *Comput. Phys. Commun.* **147**, 530 (2002).
- [4] O Métais, in *New trends in turbulence: Les Houches lectures* edited by M Lesieur, A Yaglom and F David (Springer-Verlag, Berlin, 2001) pp. 113–139
- [5] D Forster, D R Nelson and M J Stephen, *Phys. Rev.* **A16**, 732 (1977)
- [6] V Yakhot and S A Orszag, *J. Sci. Comput.* **1**, 3 (1986)
- [7] W D McComb, *The physics of fluid turbulence* (Oxford University Press, Clarendon, 1990)
- [8] W D McComb and A G Watt, *Phys. Rev.* **A46**, 4797 (1992)
- [9] Y Zhou, G Vahala and M Hussain, *Phys. Rev.* **A37**, 2590 (1988)
- [10] Y Zhou and G Vahala, *Phys. Rev.* **E47**, 2503 (1993)



- [11] R H Kraichnan, *J. Atmos. Sci.* **33**, 1521 (1976)
- [12] O Schilling and Y Zhou, *Phys. Fluids* **14**, 2002 (2002)
- [13] M Lesieur, *Turbulence in fluids – Stochastic and numerical modelling* (Kluwer Academic Publishers, Dordrecht, 1990).
- [14] D Carati and L Brenig, *Phys. Rev.* **A40**, 5193 (1989)
- [15] M K Verma, *Pramana – J. Phys.* **61**, 577 (2003)
- [16] S A Orszag and G S Patterson, *Phys. Rev. Lett.* **28**, 76 (1972)
- [17] M Frigo and S G Johnson, [www.fftw.org](http://www.fftw.org)
- [18] J Mathieu and J Scott, *The introduction to turbulent flow* (Cambridge University Press, Cambridge, 2000)
- [19] O Agullo, W-C Müller, B Knaepen and D Carati, *Phys. Plasma* **8**, 3502 (2001)
- [20] W-C Müller and D Carati, *Phys. Plasma* **9**, 824 (2002)
- [21] W-C Müller and D Carati, *Comput. Phys. Commun.* **147**, 544 (2002)
- [22] Y Zhou, O Schilling and S Ghosh, *Phys. Rev.* **E66**, 26309 (2002)
- [23] G Dar, M K Verma and V Eswaram, *Physica* **D157**, 207 (2001)