Field theoretic calculation of renormalized viscosity, renormalized resistivity, and energy fluxes of magnetohydrodynamic turbulence

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A self-consistent renormalization scheme has been applied to nonhelical magnetohydrodynamic (MHD) turbulence with zero cross helicity. Kolmogorov's 5/3 power law has been shown to be a consistent solution for $d \geqslant d_c \approx 2.2$. For Kolmogorov's solution, both renormalized viscosity and resistivity are positive for the whole range of parameters. Various cascade rates and Kolmogorov's constant for MHD turbulence have been calculated by solving the flux equation to first order in the perturbation series. We find that the magnetic energy cascades forward. The Kolmogorov constant for $d=3$ does not vary significantly with r_A and is found to be close to the constant for fluid turbulence.

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The statistical theory of magnetohydrodynamic (MHD) turbulence is one of the important problems of current research. The quantities of interests in this area are the energy spectrum, cascade rates, intermittency exponents, etc. In this article we analytically compute the renormalized viscosity, renormalized resistivity, and cascade rates using field theoretic techniques.

The incompressible MHD equation in Fourier space is given by

$$(-i\omega+\nu k^2)u_i(\hat{k}) = -\frac{i}{2}P^{+}_{ijm}(\mathbf{k})\int d\hat{\mathbf{p}}[u_j(\hat{p})u_m(\hat{k}-\hat{p}) - b_j(\hat{\mathbf{p}})b_m(\hat{\mathbf{k}}-\hat{\mathbf{p}})], \tag{1}$$

$$(-i\omega+\lambda k^2)b_i(\hat{k}) = -iP^{-}_{ijm}(\mathbf{k})\int d\hat{\mathbf{p}}[u_j(\hat{p})b_m(\hat{k}-\hat{p})], \tag{2}$$

$$k_i u_i(\mathbf{k}) = 0, \tag{3}$$

$$k_i b_i(\mathbf{k}) = 0, \tag{4}$$

where $\mathbf{u}$ and $\mathbf{b}$ are the velocity and magnetic field fluctuations, respectively, ν and λ are the viscosity and the resistivity, respectively, and d is the space dimension. Also,

$$P^{+}_{ijm}(\mathbf{k}) = k_j P_{im}(\mathbf{k}) + k_m P_{ij}(\mathbf{k}), \tag{5}$$

$$P_{im}(\mathbf{k}) = \delta_{im} - \frac{k_i k_m}{k^2}, \tag{6}$$

$$P^{-}_{ijm}(\mathbf{k}) = k_j \delta_{im} - k_m \delta_{ij}, \tag{7}$$

$$\hat{k} = (\mathbf{k},\boldsymbol{\omega}), \quad d\hat{p} = d\mathbf{p}\,d\omega/(2\pi)^{d+1}. \tag{8}$$

The energy spectra for MHD, $E^u(k)$ and $E^b(k)$, are still under debate. Kraichnan [1] and Iroshnikov [2] first gave the phenomenology of steady-state, homogeneous, and isotropic MHD turbulence, and proposed that the spectrum is proportional to $k^{-3/2}$. Later Marsch [3], Matthaeus and Zhou [4], and Zhou and Matthaeus [5] proposed an alternative phenomenology in which the energy spectra are proportional to $k^{-5/3}$, similar to Kolmogorov's spectrum for fluid turbulence. Current numerical [6–8] and theoretical [9–11] work supports Kolmogorov-like phenomenology for MHD turbulence. In the present paper we show that Kolmogorov's spectrum ($\propto k^{-5/3}$) is a consistent solution of the renormalization group (RG) equation of MHD turbulence.

Forster *et al.*, DeDominicis and Martin, Fournier and Frisch, and Yakhot and Orszag [12] applied the RG technique to fluid turbulence. They considered external forcing and calculated renormalized parameters: viscosity, noise coefficient, and vertex. McComb [13] instead applied a self-consistent RG procedure; here the energy spectrum was assumed to be Kolmogorov's power law, and the renormalized viscosity was computed iteratively. For MHD turbulence, Fournier *et al.*, Camargo and Tasso, and Liang and Diamond [14] employed the RG technique on similar lines as that of Forster *et al.* [12]. In this article we will apply McComb's self-consistent technique to MHD turbulence. Earlier Verma [9] did a self-consistent calculation and showed that the mean magnetic field gets renormalized, and the Kolmogorov power law is a consistent solution of the MHD RG equation. Here we will carry out the renormalization of viscosity and resistivity.

For simplicity of the calculation we assume that the mean magnetic field is absent. This allows us to assume the turbulence to be isotropic to a reasonable approximation. In the presence of a mean magnetic field, turbulence becomes anisotropic; this issue has been studied by Sridhar and Goldreich [10] and Goldreich and Sridhar [11]. In addition to the above assumption, we also take cross helicity ($2\mathbf{u}\cdot\mathbf{b}$), magnetic helicity ($\mathbf{a}\cdot\mathbf{b}$), and kinetic helicity ($\mathbf{u}\cdot\boldsymbol{\omega}$) to be zero, where $\mathbf{a}$ is the magnetic vector potential, and $\boldsymbol{\omega}$ is the vorticity.

In our RG procedure the wave number range (k_N, k_0) is divided logarithmically into N shells. We carry out the elimination of the first shell $k^{>} = (k_1, k_0)$ and obtain the modified MHD equation for $k^{<} = (k_N, k_1)$. This process is continued

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for all the shells. The shell elimination is performed by ensemble averaging over $k^>$ modes [12,14]. We assume that $u_i^>(\hat{k})$, and $b_i^>(\hat{k})$ have Gaussian distributions with zero mean, while $u_i^<(\hat{k})$ and $b_i^<(\hat{k})$ are unaffected by the averaging process. In addition we take

$$\langle u_i^>(\hat{p}) u_j^>(\hat{q})\rangle = P_{ij}(\mathbf{p}) C^{uu}(\hat{p}) \delta(\hat{p}+\hat{q}), \tag{9}$$

$$\langle b_i^>(\hat{p}) b_j^>(\hat{q})\rangle = P_{ij}(\mathbf{p}) C^{bb}(\hat{p}) \delta(\hat{p}+\hat{q}). \tag{10}$$

Let us denote by $\nu_{(n)}$ and $\lambda_{(n)}$ the viscosity and resistivity after the elimination of the n shell. To first order of perturbation, we obtain

$$(-i\omega + \nu_{(n)}k^2 + \delta\nu_{(n)}k^2) u_i^<(\hat{k}) = -\frac{i}{2} P_{ijm}^+(\mathbf{k}) \int d\hat{p} [u_j^<(\hat{p}) u_m^<(\hat{k}-\hat{p}) - b_j^<(\hat{p}) b_m^<(\hat{k}-\hat{p})], \tag{11}$$

$$(-i\omega + \lambda_{(n)}k^2 + \delta\lambda_{(n)}k^2) b_i^<(\hat{k}) = -i P_{ijm}^-(\mathbf{k}) \int d\hat{p} [u_j^<(\hat{p}) b_m^<(\hat{k}-\hat{p})], \tag{12}$$

where

$$\delta\nu_{(n)}(k) = \frac{1}{(d-1)k^2} \int_{\hat{\mathbf{p}}+\hat{\mathbf{q}}=\hat{\mathbf{k}}}^{\Delta} \frac{d\mathbf{p}}{(2\pi)^d} \times \left[S(k,p,q) \frac{C^{uu}(q)}{\nu_{(n)}(p)p^2 + \nu_{(n)}(q)q^2} - S_6(k,p,q) \frac{C^{bb}(q)}{\lambda_{(n)}(p)p^2 + \lambda_{(n)}(q)q^2} \right], \tag{13}$$

$$\delta\lambda_{(n)}(k) = \frac{1}{(d-1)k^2} \int_{\hat{\mathbf{p}}+\hat{\mathbf{q}}=\hat{\mathbf{k}}}^{\Delta} \frac{d\mathbf{p}}{(2\pi)^d} \times \left[-S_8(k,p,q) \frac{C^{bb}(q)}{\nu_{(n)}(p)p^2 + \lambda_{(n)}(q)q^2} + S_9(k,p,q) \frac{C^{uu}(q)}{\lambda_{(n)}(p)p^2 + \nu_{(n)}(q)q^2} \right] \tag{14}$$

with $S_i(k,p,q)$ as functions of k,p, and q. Hence, after the elimination of the $(n+1)$th shell, the effective viscosity and resistivity will be $(\nu,\lambda)_{(n+1)}(k) = (\nu,\lambda)_{(n)}(k) + \delta(\nu,\lambda)_{(n)}(k)$.

We solve the above equations iteratively. To simplify, we replace $C(k)$ in Eqs. (13) and (14) by the one-dimensional energy spectrum $E(k)$,

$$C^{uu,bb}(k) = \frac{2(2\pi)^d}{S_d(d-1)} k^{-(d-1)} E^{u,b}(k), \tag{15}$$

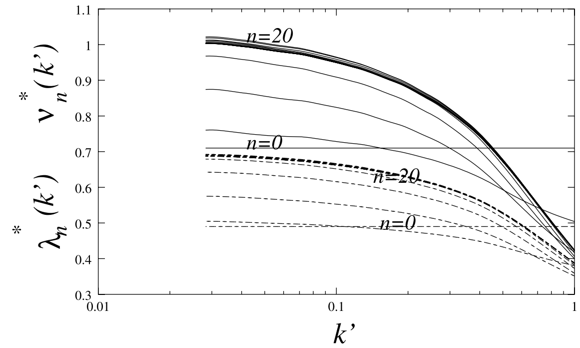

FIG. 1. Plot of $\nu^*(k')$ (solid lines) and $\lambda^*(k')$ (dashed lines) vs k' for $d=3$ and $\sigma_c=0$, $r_A=1$. Values at various iterations are shown by different curves.

where S_d is the surface area of d-dimensional spheres. We assume that $E^u(k)$ and $E^b(k)$ follow

$$E^u(k) = K^u \Pi^{2/3} k^{-5/3}, \quad E^b(k) = E^u(k)/r_A. \tag{16}$$

Regarding $\nu_{(n)}$ and $\lambda_{(n)}$, we attempt the following form of solution:

$$(\nu,\lambda)_{(n)}(k_n k') = (K^u)^{1/2} \Pi^{1/3} k_n^{-4/3} (\nu,\lambda)^*_{(n)}(k') \tag{17}$$

with $k = k_{n+1}k' (k'<1)$ with the expectation that $\nu^*_{(n)}(k')$ and $\lambda^*_{(n)}(k')$ are universal functions for large n. We numerically solve for $\nu^*_{(n)}(k')$ and $\lambda^*_{(n)}(k')$. Our calculations reveal that the solutions of $\nu^*_{(n)}(k')$ and $\lambda^*_{(n)}(k')$ converge for all $d > d_c \approx 2.2$. From this observation we can conclude that Kolmogorov's energy spectrum $[E(k) \propto k^{-5/3}]$ is a consistent solution of the RG equations. Meanwhile, Kraichnan's 3/2 energy spectrum and $\nu k^2 = \lambda k^2 \propto k B_0$, where B_0 (a constant) is the magnetic field of the large eddies, do not satisfy the renormalization group equations [Eqs. (13,14)]. Hence $E(k) \propto k^{-3/2}$ is not a consistent solution of the RG equations. Our result regarding the nonexistence of a stable RG fixed point for $d=2$ is consistent with the RG calculation of Liang and Diamond [14]. Refer to Fig. 1 for illustration of $\nu^*_{(n)}(k')$ and $\lambda^*_{(n)}(k')$ for $d=3$ and $r_A=1$.

The values of renormalized parameters for $d=3$ and various r_A are shown in Table I. For large r_A, the asymptotic ν^* is close to the corresponding value for fluid turbulence, but the asymptotic λ^* is also comparable to ν^*. This implies that in the fluid dominated regime there is a significant magnetic energy flux in addition to the usual Kolmogorov flux in fluid modes. As r_A is decreased, ν^* increases but λ^* decreases. This trend is seen until $r_A \approx 0.25$, where the RG fixed point with nonzero ν^* and λ^* becomes unstable, and the trivial RG fixed point with $\nu^* = \lambda^* = 0$ becomes stable. This result suggests an absence of turbulence for r_A below 0.25. This is consistent with the fact that the MHD equations become linear in the $r_A \to 0$ (fully magnetic) limit.

We can proceed further and compute various cascade rates and Kolmogorov's constant for MHD using the renormalized parameters computed above. To compute these quantities we resort to the energy equations, which are [15,16]

TABLE I. The values of ν^*, λ^*, ν^{uu*}, ν^{ub*}, λ^{bu*}, and λ^{bb*} for various r_A when $d=3$ and $\sigma_c=0$.

r_A	ν^*	λ^*	$\Pi^{u<}_{u>}/\Pi$	$\Pi^{u<}_{b>}/\Pi$	$\Pi^{b<}_{u>}/\Pi$	$\Pi^{b<}_{b>}/\Pi$	$\Pi^{u<}_{b<}/\Pi$	K
∞	0.38		1					1.53
5000	0.36	0.85	1	3.5×10^{-4}	-1.05×10^{-4}	2.4×10^{-4}	1.3×10^{-4}	1.51
5	0.47	0.82	0.61	0.26	-0.050	0.19	0.13	1.51
1	1.00	0.69	0.12	0.39	0.12	0.37	0.49	1.50
0.5	2.11	0.50	0.037	0.33	0.33	0.30	0.63	1.65
0.3	11.0	0.14	0.011	0.36	0.42	0.21	0.63	3.26
0.2	–	–	–	–	–	–	–	–

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)C^{uu}(\mathbf{k},t)=\frac{1}{(d-1)(2\pi)^d\delta(\mathbf{k}+\mathbf{k}')}\times\int_{\mathbf{k}'+\mathbf{p}+\mathbf{q}=\mathbf{0}}\frac{d\mathbf{p}}{(2\pi)^d}[S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})+S^{uu}(\mathbf{k}'|\mathbf{q}|\mathbf{p})+S^{ub}(\mathbf{k}'|\mathbf{p}|\mathbf{q})+S^{ub}(\mathbf{k}'|\mathbf{q}|\mathbf{p})], \quad (18)$$

$$\left(\frac{\partial}{\partial t}+2\lambda k^2\right)C^{bb}(\mathbf{k},t)=\frac{1}{(d-1)(2\pi)^d\delta(\mathbf{k}+\mathbf{k}')}\times\int_{\mathbf{k}'+\mathbf{p}+\mathbf{q}=\mathbf{0}}\frac{d\mathbf{p}}{(2\pi)^d}[S^{bu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})+S^{bu}(\mathbf{k}'|\mathbf{q}|\mathbf{p})+S^{bb}(\mathbf{k}'|\mathbf{p}|\mathbf{q})+S^{bb}(\mathbf{k}'|\mathbf{q}|\mathbf{p})], \quad (19)$$

where

$$S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})=-\mathrm{Im}\{[\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})][\mathbf{u}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{p})]\}, \quad (20)$$

$$S^{bb}(\mathbf{k}'|\mathbf{p}|\mathbf{q})=-\mathrm{Im}\{[\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})][\mathbf{b}(\mathbf{k}')\cdot\mathbf{b}(\mathbf{p})]\}, \quad (21)$$

$$S^{ub}(\mathbf{k}'|\mathbf{p}|\mathbf{q})=\mathrm{Im}\{[\mathbf{k}'\cdot\mathbf{b}(\mathbf{q})][\mathbf{u}(\mathbf{k}')\cdot\mathbf{b}(\mathbf{p})]\}, \quad (22)$$

$$S^{bu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})=-S^{ub}(\mathbf{p}|\mathbf{k}'|\mathbf{q}). \quad (23)$$

Here Im stands for the imaginary part of the argument, and the above integrals have the constraints that $\mathbf{k}'+\mathbf{p}+\mathbf{q}=\mathbf{0}$ ($\mathbf{k}=-\mathbf{k}'$). The energy equations in the above form were written by Dar *et al.* [16], who interpret the terms $S(\mathbf{k}|\mathbf{p}|\mathbf{q})$ as the energy transfer rate from mode $\mathbf{p}$ (second argument of S) to $\mathbf{k}$ (first argument of S) with mode $\mathbf{q}$ (third argument of S) acting as a mediator. This interpretation of energy transfer due to Dar *et al.* [16] is consistent with the earlier formalism.

We can derive an expression for the energy transfer rate or energy flux from a wavenumber sphere using $S(\mathbf{k}'|\mathbf{p}|\mathbf{q})$. The formula for the energy flux from inside the X sphere ($X<$) to the outside of the Y sphere ($Y>$) is

$$\Pi^{X<}_{Y>}(k_0)=\int_{k>k_0}\frac{d\mathbf{k}}{(2\pi)^d}\int_{p<k_0}\frac{d\mathbf{p}}{(2\pi)^d}\langle S^{YX}(\mathbf{k}'|\mathbf{p}|\mathbf{q})\rangle, \quad (24)$$

where X and Y stand for u or b. In our study we assume that the kinetic energy is forced at small wave numbers, and the turbulence is steady. We calculate the above fluxes analytically to leading order in the perturbation series using the same procedure as that of Leslie [17]. The flux is calculated using Eq. (24) by taking the ensemble average of S^{YX}. The expression for $\langle S^{bb}(k|p|q)\rangle$ is

$$\langle S^{bb}(k|p|q)\rangle=\int_{-\infty}^{t}dt'[T_4(k,p,q)G^{bb}(k,t-t')\times C^{bb}(p,t-t')C^{uu}(q,t-t')+T_8(k,p,q)G^{bb}(p,t-t')\times C^{bb}(k,t-t')C^{uu}(q,t-t')+T_{10}(k,p,q)G^{uu}(q,t-t')C^{bb}(k,t-t')\times C^{bb}(p,t-t')], \quad (25)$$

where $T_i(k,p,q)$ are functions of wave vectors k,p, and q. The expressions for other transfer rates $\langle S^{uu}(k|p|q)\rangle$, $\langle S^{ub}(k|p|q)\rangle$, and $\langle S^{bu}(k|p|q)\rangle$ look similar. In the above formulas we substitute Kolmogorov's spectrum [Eqs. (16)] for the energy spectrum, and the following expression for the effective viscosity and resistivity:

$$(\nu,\lambda)(k)=(K^u)^{1/2}\Pi^{1/3}k^{-4/3}(\nu^*,\lambda^*) \quad \text{for} \quad k\geq k_n. \quad (26)$$

Following the same procedure as Leslie [17], we obtain the following nondimensional form of the equations:

$$\frac{\Pi^{X<}_{Y>}}{\Pi}=\frac{4S_{d-1}}{(d-1)^2S_d}(K^u)^{3/2}\int_0^1 dv\,\ln(1/v)\times\int_{1-v}^{1+v}dw(vw)^{d-2}(\sin\alpha)^{d-3}F^{X<}_{Y>} \quad (27)$$

where the integrals $F^{X<}_{Y>}$ are functions of v, w, ν^*, and λ^*. After a bit of manipulation we can obtain $\Pi^{X<}_{Y>}/\Pi$ and the constant K^u. In addition we can also obtain Kolmogorov's constant K for the total energy,

$$E(k)=K\Pi^{2/3}k^{-5/3}, \quad (28)$$

using $K=K^u(1+r_A^{-1})$. The values of $\Pi^{X<}_{Y>}/\Pi$ and K for $d=3$ and various r_A are listed in Table I.

The entries in Table I show that the cascade rates $\Pi^{u<}_{b>}$, $\Pi^{b<}_{b>}$, $\Pi^{u<}_{b<}$, $\Pi^{b<}_{u>}$ are approximately of the same order for r_A between 0.5 and 1, but the flux $\Pi^{u<}_{u>}$ is rather small. The sign of $\Pi^{b<}_{b>}$ is positive, indicating that the magnetic energy ME cascades forward, that is, from large length scales to small length scales. The magnetic energy thus appearing at small length scales will be lost due to resistive dissipation, and the large-scale magnetic field is maintained by the $\Pi^{u<}_{b<}$ flux. The Kolmogorov constant K is approximately constant and is close to 1.6, the same as that for fluid turbulence ($r_A=\infty$), for all r_A greater than 0.5.

To summarize, we employed a self-consistent RG scheme for MHD turbulence and found that Kolmogorov's 5/3 power law is a consistent solution of the RG equations for $d \geqslant d_c \approx 2.2$. For Kolmogorov's solution, the renormalized viscosity and resistivity have been calculated, and they are found to be positive. For $d=3$, variation of ν^* and λ^* with r_A shows some interesting features. For large r_A, ν^* is the same as that for fluid turbulence, but λ^* is also nonzero, in fact larger than ν^*. As r_A is decreased, ν^* increases but λ^* decreases until $r_A \approx 0.25$ at which value turbulence disappears.

Using the flux equations we have obtained various fluxes and Kolmogorov's constant K. For $d=3$, K does not vary significantly with the variation of r_A, and it is close to K for fluid turbulence. We find that the cascade rate from the magnetic sphere to outside the magnetic sphere ($\Pi^{b<}_{b>}$) is positive, a result consistent with the numerical results of Dar *et al.* [16].

In this paper we have restricted ourselves to nonhelical turbulence. Helical MHD turbulence is very important especially in the light of enhancement of magnetic energy (dynamo). However, the physics of helical turbulence is more complex with the appearance of an inverse cascade of magnetic helicity, etc. The field theoretic analysis for this case will be taken up later. Recent studies show that the mean magnetic field has a strong effect on the energy spectrum, and it induces anisotropy. A full-fledged field theory calculation in the presence of a mean magnetic field is also necessary for a clearer picture of MHD turbulence.

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