Mean magnetic field renormalization and Kolmogorov's energy spectrum in magnetohydrodynamic turbulence

Mahendra K. Verma[a)]
Department of Physics, Indian Institute of Technology, Kanpur-208016, India

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A self-consistent renormalization group procedure has been constructed for magnetohydrodynamic turbulence in which small wave number modes are averaged out, and the effective mean magnetic field at large wave numbers is obtained self consistently. In this renormalization group scheme, it is found that an $E(k)$ proportional to $k^{-5/3}$ (Kolmogorov's spectrum) is a self-consistent solution, and the procedure yields a self-consistent effective mean magnetic field proportional to $k^{-1/3}$. It is also deduced from the formalism that the magnitude of the cascade rate decreases as the strength of the mean magnetic field is increased.
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I. INTRODUCTION

Kolmogorov hypothesized that the energy spectrum $E(k)$ of fluid turbulence in the inertial range is isotropic and is a power law with a spectral index of $-5/3$, i.e.,

$$E(k)=K_{K0}\Pi^{2/3}k^{-5/3}, \tag{1}$$

where K_{Ko} is an universal constant called Kolmogorov's constant, k is the wave number, and Π is the nonlinear energy cascade rate. Note that Π is equal to the dissipation rate and also the energy supply rate of the fluid. Experiments,[1] simulations,[2] and some of the analytical calculations based on the direct interaction approximation,[3,4] renormalization group (RG) techniques,[5–11] self-consistent mode coupling,[12] etc. are in good agreement with the above phenomenology.

In this paper, we will discuss the energy spectrum in magnetohydrodynamic (MHD) turbulence. In MHD there are two fields, the velocity field $\mathbf{u}$ and the magnetic field $\mathbf{B}=\mathbf{B}_0+\mathbf{b}$, where $\mathbf{B}_0$ is the mean magnetic field or the magnetic field of the large eddies, and $\mathbf{b}$ is the magnetic field fluctuation. It is also customary to use Elsässer variables $\mathbf{z}^{\pm}=\mathbf{u}\pm\mathbf{b}$. Here the magnetic field has been written in velocity units ($b/\sqrt{4\pi\rho}$, where ρ is the density of the fluid). We also assume that the plasma is incompressible.

There are two time scales in magnetofluid: (i) nonlinear time scale $1/(kz_k^{\pm})$ (similar to that in fluid turbulence) and (ii) Alfvén time scale $1/(kB_0)$. Kraichnan,[13] Iroshnikov,[14] and Dobrowolny *et al.*[15] argued that the interacting z_k^+ and z_k^- modes will get separated in one Alfvén time scale because of the mean magnetic field. Therefore, they chose the Alfvén time scale $\tau_A=(kB_0)^{-1}$ as the relevant time scale and found that

$$\Pi^+\approx\Pi^-\approx\frac{1}{B_0}E^+(k)E^-(k)k^3=\Pi, \tag{2}$$

where $\Pi^{\pm}$ are the cascade rates of $z_k^{\pm}$. If $E^+(k)\approx E^-(k)$, then the above equation implies that

$$E^+(k)\approx E^-(k)\approx(B_0\Pi)^{1/2}k^{-3/2}. \tag{3}$$

In absence of a mean magnetic field, the magnetic field of the largest eddy was taken as B_0. Kraichnan[13] also argued that the fluid and magnetic energies are equipartitioned. The above phenomenology is referred to as Dobrowolny *et al.*'s generalized Kraichnan–Iroshnikov (KID) phenomenology.

If the nonlinear time scale $\tau_{NL}^{\pm}\approx(kz_k^{\mp})^{-1}$ is chosen as the interaction time scales for the eddies $z_k^{\pm}$, we obtain

$$\Pi^{\pm}\approx(z_k^{\pm})^2(z_k^{\mp})k, \tag{4}$$

which in turn leads to

$$E^{\pm}(k)=K^{\pm}(\Pi^{\pm})^{4/3}(\Pi^{\mp})^{-2/3}k^{-5/3}, \tag{5}$$

where $K^{\pm}$ are constants, which we will refer to as Kolmogorov's constants for MHD turbulence. Because of its similarity to Kolmogorov's fluid turbulence phenomenology, this phenomenology is referred to as Kolmogorov-like MHD turbulence phenomenology. This phenomenology was first given by Marsch,[16] Matthaeus and Zhou,[17] and Zhou and Matthaeus[18] (it is a limiting case of a more generalized phenomenology constructed by Matthaeus and Zhou,[17] and Zhou and Matthaeus[18]). It is implicit in these phenomenological arguments that KID phenomenology is expected to hold when $B_0\gg\sqrt{kE^{\pm}(k)}$, while Kolmogorov-like phenomenology is expected to be applicable when $B_0\ll\sqrt{kE^{\pm}(k)}$.

In the solar wind, which is a good testing ground for MHD turbulence theories, Matthaeus and Goldstein[19] found that the exponents of the total energy and magnetic energy are 1.69 ± 0.08 and 1.73 ± 0.08, respectively, somewhat closer to 5/3 than 3/2. This is more surprising because $B_0\gg\sqrt{kE^{\pm}(k)}$ for inertial range wavenumbers in the solar wind. The numerical simulations also tend to indicate that the Kolmogorov-like phenomenology, rather than KID phenomenology, is probably applicable in MHD turbulence.[20] Hence, the comparison of the solar wind observations and simulation results with the phenomenological predictions appears to show that there are some inconsistencies in the phenomenological arguments given above. To resolve these in-

[a)]Electronic mail: mkv@iitk.ac.in

consistencies, we have attempted to examine the MHD equations using renormalization group analysis.

For fluid turbulence, Forster *et al.*[5] and Yakhot and Orszag[6] have applied dynamical RG procedure in which a forcing term with a power law distribution in wave number space is introduced. McComb,[8] McComb and Shanmugasundaram,[9] McComb and Watt,[10] and Zhou *et al.*[11] instead applied a self-consistent RG procedure that yields Kolmogorov's energy spectrum. For MHD turbulence, Fournier *et al.*[21] and Camargo and Tasso[22] have used a RG procedure similar to that of Forster *et al.*[5] and Yakhot and Orszag.[6] In all these schemes, the averaging is done over the small scales (based on Wilson's approach in his Fourier space RG). To date, the RG methods applied to MHD turbulence do not find direct evidence of Kolmogorov-like power law in MHD turbulence. In a more recent work, Verma and Bhattacharjee[23] have applied Kraichnan's DIA[3,4] to MHD turbulence and obtained the Kolmogorov's constant for MHD, but in Verma and Bhattacharjee's work $k^{-5/3}$ energy spectra was assumed, and an artificial cutoff was introduced for the self-energy integral.

In this paper, we construct a self-consistent RG procedure similar to that used by McComb,[8] McComb and Shanmugasundaram,[9] McComb and Watt,[10] and Zhou *et al.*[11] for fluid turbulence. However, one major difference is that we integrate the small wave number modes instead of integrating the large wave number modes, as done by earlier authors. In our procedure, we obtain the effective mean magnetic field $B_0(k)$ as we go from small wave numbers to large wave numbers. At small wave numbers, the MHD equations are approximately linear. During the RG process, the effects of the nonlinear terms in the small wave number shells are translated to the modification of $B_0(k)$ at larger wave numbers.

We postulate that the effective mean magnetic field is the magnetic field of the next-largest eddy contrary to the KID phenomenology where the effective mean magnetic field at any scale is a constant. To illustrate, for Alfvén waves of wave number k, the effective magnetic field $B_i(k)$ (after ith iteration of the RG procedure defined below) will be the magnetic field of the eddy of size $k/10$ or so. This argument is based on the physical intuition that for the scattering of the Alfvén waves at a wave number k, the effects of the magnetic field of the next-largest eddy is much more than that of the external field. The mean magnetic field at the largest scale will simply convect the waves, whereas the local inhomogeneities contribute to the scattering of waves, which leads to turbulence (note that in WKB method, the local inhomogeneity of the medium determines the amplitude and the phase evolution). In our scheme, we show that $E(k)\propto k^{-5/3}$ and the mean magnetic field proportional to $k^{-1/3}$ are the self-consistent solutions of the RG equations. Thus we argue that B_0 appearing in the KID's phenomenology must be k dependent. Note that the substitution of k dependent $B_0(k)$ in Eq. (3) yields $k^{-5/3}$ energy spectrum, which is consistent with the solar wind observations and the simulation results. We will describe these ideas in more detail in the following section.

The normalized cross helicity σ_c, defined as $(E^+ - E^-)/(E^+ + E^-)$, and the Alfvén ratio r_A, defined as the ratio of fluid energy and magnetic energy, are important factors in MHD turbulence. For simplicity of the calculation, we have taken $E^+(k)=E^-(k)$ and $r_A=1$. These conditions are met at many places in the solar wind and in other astrophysical plasmas.

II. CALCULATION

The MHD equation in the Fourier space is[13]

$$(-i\omega \mp i(\mathbf{B}_0\cdot\mathbf{k}))z_i^{\pm}(\mathbf{k},\omega) = -iM_{ijm}(\mathbf{k})\int d\mathbf{p}\,d\omega' z_j^{\mp}(\mathbf{p},\omega')z_m^{\pm}(\mathbf{k}-\mathbf{p},\omega-\omega'), \tag{6}$$

where

$$M_{ijm}(\mathbf{k})=k_jP_{im}(\mathbf{k}); \quad P_{im}(\mathbf{k})=\delta_{im}-\frac{k_ik_m}{k^2}. \tag{7}$$

Here, we have ignored the viscous terms. The above equation will, in principle, yield an anisotropic energy spectra (different spectra along and perpendicular to $\mathbf{B}_0$). Since the anisotropic equation is quite complicated to solve using RG, we modify the above equation to the following form to preserve isotropy:

$$(-i\omega \mp i(B_0k))z_i^{\pm}(\mathbf{k},\omega) = -iM_{ijm}(\mathbf{k})\int d\mathbf{p}\,d\omega' z_j^{\mp}(\mathbf{p},\omega')z_m^{\pm}(\mathbf{k}-\mathbf{p},\omega-\omega'). \tag{8}$$

This equation can be thought of as an effective MHD equation in an isotropically random mean magnetic field.

In our RG procedure, the wave number range $(k_0..k_N)$ is divided logarithmically into N shells. The nth shell is $(k_{n-1}..k_n)$, where $k_n=s^nk_0(s>1)$. The modes in the first few shells will be the energy containing eddies that will force the turbulence. To keep our calculation procedure simple, we assume that the external forcing maintains the energy of the first few shells to the initial values. The modes in the first few shells are assumed to be random with a gaussian distribution.

In the following discussion, we first carry out the elimination of the first shell $(k_0..k_1)$ and obtain the modified MHD equation. We then proceed iteratively to eliminate higher shells and get a general expression for the modified MHD equation after elimination of the nth shell. The details of the renormalization group operation are as follows.

A. RG procedure

1. Decompose the modes into the modes to be eliminated $(k^<)$ and the modes to be retained $(k^>)$. In the first iteration, $(k_0..k_1)=k^<$ and $(k_1..k_N)=k^>$. Note that $B_0(k)$ is the mean magnetic field before the elimination of the first shell.

2. We rewrite the Eq. (8) for $k^<$ and $k^>$. The equation for $z_i^{\pm>}(\mathbf{k},t)$ modes is

$$(-i\omega \mp i(B_0k))z_i^{\pm>}(\mathbf{k},\omega) = -iM_{ijm}(\mathbf{k})\int d\mathbf{p}d\omega'[z_j^{\mp>}(\mathbf{p},\omega')z_m^{\pm>}(\mathbf{k}-\mathbf{p},\omega-\omega')] + [z_j^{\mp>}(\mathbf{p},\omega')z_m^{\pm<}(\mathbf{k}-\mathbf{p},\omega-\omega')$$
$$+z_j^{\mp<}(\mathbf{p},\omega')z_m^{\pm>}(\mathbf{k}-\mathbf{p},\omega-\omega')] + [z_j^{\mp<}(\mathbf{p},\omega')z_m^{\pm<}(\mathbf{k}-\mathbf{p},\omega-\omega')], \tag{9}$$

while the equation for $z_i^{\pm<}(\mathbf{k},t)$ modes can be obtained by interchanging $<$ and $>$ in the above equation.

3. The terms given in the second and third brackets in the right hand side (RHS) of Eq. (9) is calculated perturbatively. We perform ensemble average over the first shell, which is to be eliminated. We assume that $z_i^{\pm<}(\mathbf{k},t)$ has a gaussian distribution with zero mean. Hence,

$$\langle z_i^{\pm<}(\mathbf{k},t)\rangle = 0, \quad \langle z_i^{\pm>}(\mathbf{k},t)\rangle = z_i^{\pm>}(\mathbf{k},\omega), \tag{10}$$

and

$$\langle z_s^{a<}(\mathbf{p},\omega')z_m^{b<}(\mathbf{q},\omega'')\rangle$$
$$= P_{sm}(\mathbf{p})C^{ab}(p,\omega')\,\delta(\mathbf{p}+\mathbf{q})\,\delta(\omega'+\omega''), \tag{11}$$

where $a,b=\pm$. Also, the triple order correlations $\langle z_s^{\pm<}(\mathbf{k},\omega)z_m^{\pm<}(\mathbf{p},\omega')z_t^{\pm<}(\mathbf{q},\omega'')\rangle$ are zero. We keep only the nonvanishing terms to first order. For the relevant Feynmann diagrams, refer to Zhou *et al.*[11] Taking $r_A=1$ and $E^+(k)=E^-(k)$, Eq. (9) becomes

$$(-i\omega \mp i(B_0k))z_i^{\pm>}(\mathbf{k},\omega) = -iM_{ijm}(\mathbf{k})\int d\mathbf{p}d\omega'[z_j^{\mp>}(\mathbf{p},\omega')z_m^{\pm>}(\mathbf{k}-\mathbf{p},\omega-\omega')]$$
$$+(-i)^2M_{ijm}(\mathbf{k})\int_{\mathbf{p}+\mathbf{q}=\mathbf{k}}^{\Delta} d\mathbf{q}d\omega' M_{mst}(\mathbf{p})P_{js}(\mathbf{q})G^{\pm\pm}(\mathbf{p},\omega')C^{\mp\mp<}(\mathbf{q},\omega-\omega')z_t^{\pm>}(\mathbf{k},\omega)$$
$$+(-i)^2M_{ijm}(\mathbf{k})\int_{\mathbf{p}+\mathbf{q}=\mathbf{k}}^{\Delta} d\mathbf{q}d\omega' M_{mst}(\mathbf{p})P_{js}(\mathbf{q})G^{\pm\mp}(\mathbf{p},\omega')C^{\mp\mp<}(\mathbf{q},\omega-\omega')z_t^{\pm>}(\mathbf{k},\omega), \tag{12}$$

where G is the Green's function obtained from the equation

$$G^{-1}(k,\omega) = \begin{pmatrix} -i\omega - ikB_0^{++}(k) & -ikB_0^{+-}(k) \\ ikB_0^{-+}(k) & -i\omega + iKB_0^{--}(k) \end{pmatrix}. \tag{13}$$

Here the integration is done over the first shell (Δ). In deriving Eq. (12) we have neglected the contribution of the triple nonlinearity $\langle z_s^{\pm>}(\mathbf{k},\omega)z_m^{\pm>}(\mathbf{p},\omega')z_t^{\pm>}(\mathbf{q},\omega'')\rangle$. McComb,[8] McComb and Shanmugsundaram,[9] and McComb and Watt[10] have also ignored the triple nonlinearity for fluid turbulence.

4. Since $r_A=1$ and $E^+(k)=E^-(k)$, we find that $B_0^{+-}(k)=B_0^{-+}(k)$. When the nonlinearity is absent, it can be easily shown that the correlation functions $C^{\pm\pm}(k,\omega)$ have the same frequency dependence as $G^{\pm\pm}(k,\omega)$, i.e.,

$$C^{\pm\pm}(k,\omega^{\pm}) = \frac{C^{\pm\pm}(k)}{-i\omega^{\pm} \mp ikB_0^{\pm\pm}(k)}. \tag{14}$$

We assume the above relationship in our perturbative scheme, as well. Note that $C^{\pm\pm}(k)=E^{\pm\pm}(k)/(4\pi k^2)$ in three dimensions. After some manipulations, the Eq. (12) becomes

$$(-i\omega \mp i[B_0(k)+\delta B_0^{\pm\pm}(k)]k)z_i^{\pm>}(\mathbf{k},t)$$
$$\mp i\,\delta B_0^{\pm\mp}(k)z_i^{\mp>}(\mathbf{k},t)$$
$$= M_{ijm}(\mathbf{k})\int d\mathbf{p}[z_j^{\mp>}(\mathbf{p},t)z_m^{\pm>}(\mathbf{k}-\mathbf{p},t)], \tag{15}$$

where

$$\delta B_0^{\pm\pm}(k)$$
$$= -k\int_{\mathbf{p}+\mathbf{q}=\mathbf{k}}^{\Delta} d\mathbf{q}\left(\frac{E(q)}{4\pi q^2}\right)$$
$$\times\left[\frac{a_2(k,p,q)[X_0^{\pm\pm}(p)+B_0^{\pm\pm}(p)] - a_4(k,p,q)B_0^{+-}(p)}{2X_0(p)[kB_0^{\pm\pm}(k)+pX_0^{\pm\pm}(p)-qX_0^{\pm\pm}(q)]}\right] \tag{16}$$

and

$$\delta B_0^{\pm\mp}(k) = -k\int_{\mathbf{p}+\mathbf{q}=\mathbf{k}}^{\Delta} d\mathbf{q}\left(\frac{E(q)}{4\pi q^2}\right)\left[\frac{a_3(k,p,q)B_0^{+-}(p) - a_1(k,p,q)[X_0^{\pm\pm}(p)+B_0^{\pm\pm}(p)]}{2X_0^{\pm\pm}(p)[kB_0^{\pm\pm}(k)+pK_0^{\pm\pm}(p)-qX_0^{\pm\pm}(q)]}\right], \tag{17}$$

where $2k^2 a_i(k,p,q)=A_i(k,p,q)$ and $X_0^{\pm\pm}(k)=\sqrt{(B_0^{\pm\pm}(k))^2-(B_0^{\pm\mp}(k))^2}$. The terms $A_i(k,p,q)$ are given in the Appendix of Leslie[4] as $B_i(k,p,q)$. Since, $E^+=E^-$ and $r_A=1$, it is clear that $\delta B_0^{++}(k)=\delta B_0^{--}(k)$. Therefore, $B_0^{++}(k)=B_0^{--}(k)=B_0(k)$ and $X_0^{++}(k)=X_0^{--}(k)=X_0(k)$. While solving for Eqs. (16) and (17) we have postulated using dynamical scaling arguments that

$$\omega_k^{\pm}=\mp k B_0^{\pm\pm}(k). \tag{18}$$

This is equivalent to using $\omega=k^z$, except that we want to express ω in terms of Alfvén speed $B_0(k)$.
Let us denote $B_1(k)$ as the effective mean magnetic field after the elimination of the first shell. Therefore,

$$B_1(k)=B_0(k)+\delta B_0(k). \tag{19}$$

Similarly,

$$B_1^{+-}(k)=B_0^{+-}(k)+\delta B_0^{+-}(k). \tag{20}$$

5. We keep eliminating the shells one after the other by the above procedure. After $n+1$ iterations, we obtain

$$B_{n+1}^{ab}(k)=B_n^{ab}(k)+\delta B_n^{ab}(k), \tag{21}$$

where the equations for $\delta B_n^{\pm\pm}(k)$ and $\delta B_n^{\pm\mp}(k)$ are the same as Eqs. (16) and (17) except that the terms $B_0^{ab}(k)$ and $X_0^{ab}(k)$ are to be replaced by $B_n^{ab}(k)$ and $X_n^{ab}(k)$, respectively. Clearly, $B_{n+1}(k)$ is the effective mean magnetic field after the elimination of the $(n+1)$th shell.

The set of RG equations to be solved are Eqs. (16) and (17) with B_0 replaced by B_ns, and Eq. (21).

B. Solution of RG equations

To solve the Eqs. (16) and (17) with B_ns and Eq. (21), we substitute the following forms for $E(k)$ and $B_n(k)$ in the modified Eqs. (16) and (17)

$$E(k)=K\Pi^{2/3}k^{-5/3}, \tag{22}$$

$$B_n^{ab}(k_n k')=K^{1/2}\Pi^{1/3}k_n^{-1/3}B_n^{*ab}(k'), \tag{23}$$

with $k=k_{n+1}k'$ $(k'>1)$. We expect that $B_n^{*ab}(k')$ is an universal function for large n. We use $\Pi^+=\Pi^-=\Pi$ due to symmetry. After the substitution, we obtain the equations for $B_n^{*ab}(k')$ that are

$$\delta B_n^*(k')=-\int_{\mathbf{p}'+\mathbf{q}'=\mathbf{k}'} d\mathbf{q}'\left(\frac{E(q')}{4\pi q'^2}\right)\left[\frac{a_2(k',p',q')[X_n^*(sp')+B_n^*(sp')]-a_4(k',p',q')B_n^{*+-}(sp')}{2X_n^*(sp')[k'B_n^*(sk')+p'X_n^*(sp')-q'X_n^*(sq')]}\right], \tag{24}$$

$$\delta B_n^{*+-}(k')=-\int_{\mathbf{p}'+\mathbf{q}'=\mathbf{k}'} d\mathbf{q}'\left(\frac{E(q')}{4\pi q'^2}\right)\left[\frac{a_3(k',p',q')B_n^{*+-}(sp')-a_1(k',p',q')[X_n^*(sp')+B_n^*(sp')]}{2X_n^*(sp')[k'B_n^*(sk')+p'X_n^*(sp')-q'X_n^*(sq')]}\right], \tag{25}$$

where the integrals in the Eqs. (24) and (25) is performed over a region $1/s\leqslant p'$, $q'\leqslant 1$ with the constraint that $\mathbf{p}'+\mathbf{q}'=\mathbf{k}'$. The recurrence relation for B_n is

$$B_{n+1}^{*ab}(k')=s^{1/3}B_n^{*ab}(sk')+s^{-1/3}\delta B_n^{*ab}(k'). \tag{26}$$

Now we need to solve the above three equations self consistently. We use Monte Carlo technique to solve the integrals. Since the integrals are identically zero for $k'>2$, the initial $B_0^*(k_i')=B_0^{*\text{initial}}$ for $k_i'<2$ and $B_0^*(k_i')=B_0^{*\text{initial}}*(k_i'/2)^{-1/3}$ for $k_i'>2$. We take $B_0^{+-}=0$. Equations (24)–(26) are solved iteratively. We continue iterating the equations till $B_{n+1}^*(k')\approx B_n^*(k')$, that is, till the solution converges. The B_n^*s for various n ranging from 0 to 3 are shown in Fig. 1. Here, the convergence is very fast, and after $n=3-4$ iterations $B_n^*(k)$ converges to an universal function

$$f(k')=1.24*B_0^{*\text{initial}}k'^{-0.32}=B_0^{*\text{initial}}(k'/2)^{-1/3}. \tag{27}$$

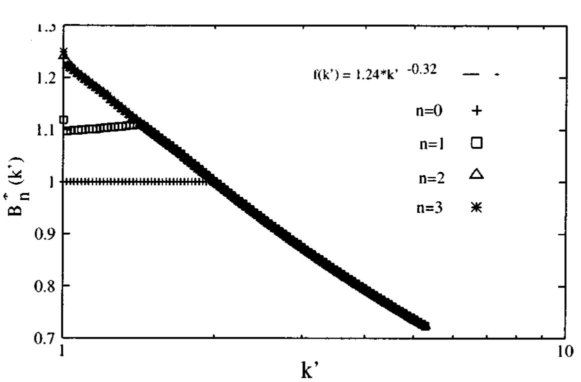

FIG. 1. $B_n^*(k')$ for $n=0...3$. The line of best fit $f(k')$ to $B_3^*(k')$ overlaps with B_3^*.

The other parameter $B_n^{*+-}(k')$ remains close to zero. The above solution of the universal function is an stable solution in the RG sense. The function B_n^* converges to the universal function $f(k')$.

The substitution of the function $B_n^*(k')$ in Eq. (23) yields

$$B_0(k)=\begin{cases} K^{1/2}\Pi^{1/2}k_0^{-1/3}B_0^{*\text{initial}}\overset{\text{def}}{=}B_0 & \text{for } k\leqslant 2k_0 \\ B_0\left(\dfrac{k}{2k_0}\right)^{-1/3} & \text{for } k>2k_0 \end{cases} \tag{28}$$

and

$$B_{n+1}(k)=K^{1/2}\Pi^{1/2}B_0^{*\text{initial}}(k/2)^{-1/3}=B_0\left(\frac{k}{2k_0}\right)^{-1/3} \tag{29}$$

for $k>k_{n+1}$ when n is large (stable RG solution). Hence, we see that $B_n(k)\propto k^{-1/3}$ in our self-consistent scheme.

To summarize, we have shown that the mean magnetic field B_0 becomes renormalized due to the nonlinear term. The self-consistent solutions of our RG schemes are

$$E(k)=K\Pi^{2/3}k^{-5/3}, \tag{30}$$

$$B_n(k)=B_0 2^{1/3}(k/k_0)^{-1/3}. \tag{31}$$

C. Calculation of *K*

We can calculate the Kolmogorov's constant for MHD turbulence K by calculating the cascade rate Π.[4] In MHD the cascade rates are

$$\Pi^{+}(k)=\Pi^{-}(k)=-\int_0^k dk' T(k'). \tag{32}$$

The numerical solution of the cascade rate integral yields[4]

$$\frac{1.24 B_0^{*\mathrm{initial}}}{K^{3/2}}=3.85. \tag{33}$$

From the above equation it is evident that the Kolmogorov's constant K is dependent on the mean magnetic field $B_0^{*\mathrm{initial}}$, in fact, $K\propto(B_0^{*\mathrm{initial}})^{2/3}$. Clearly, an increase in the mean magnetic field leads to an increase in the Kolmogorov constant, which in turn will lead to a suppression in the cascade rate [cf. Eq. (5)]. This result is consistent with the simulation results of Oughton.[24] However, a cautious remark is necessary here. We have considered the mean magnetic field to be isotropic; this isotropy assumption has to be dropped for studies of realistic situations.

III. DISCUSSIONS AND CONCLUSIONS

In this paper, we have worked out the renormalization group scheme for the MHD equations to first order. We average out the small wavenumber modes and solve for the renormalized mean magnetic field $B_n(k)$. Ours is a self-consistent scheme in which we show that $E(k)$ proportional to $k^{-5/3}$ is consistent with our procedure, and that this procedure yields a self-consistent $B_n(k)$ proportional to $k^{-1/3}$. When we start with B_0, after each RG step the mean magnetic field increases in the region $k<2k_0$ [see Eq. (29)] until we get a stable $B_n(k)$, which is proportional to $k^{-1/3}$. Clearly, the Alfvén time scale using $B_1(k)$ is less than that using $B_0(k)$. Similarly, for B_n, we find that $\tau_A^{B_{n+1}}<\tau_A^{B_n}$. The relavant nonlinear time scale is usually taken to be the lowest time scale of the system. Therefore, the scale-dependent $B_n(k)$ should be the nonlinear time scale that yields Kolomogorov-like energy spectrum for MHD. This result justifies our choice of $B_n(k)$ as the effective mean magnetic field for the analysis. Note that KID take $\tau_A\approx(kB_0)^{-1}$ to be the effective time scale for the nonlinear interactions that gives $E(k)\propto k^{-3/2}$. Also, $\tau_{NL}^{B_n}$ is of the same order as the nonlinear time scales of $z^{\pm}$, $\tau_{NL}^{\pm}\approx(kz_k^{\pm})^{-1}$. The quantity $\tau_{NL}^{B_{n+1}}$ can possibly be obtained numerically using the time evolution of the Fourier components; this test will validate the theoretical assumptions made in our paper. The numerical investigation in this direction is under progress.

The physical idea behind our argument is that for the scattering of the Alfvén waves at a wave number k, the effective magnetic field is due more to the next largest eddy, than to the mean magnetic field. In other words, the effective mean magnetic field, which appears in the Kraichnan–Iroshnokov–Dobrowolny (KID) turbulence phenomenology, should really be the renormalized mean magnetic field (an scale dependent quantity). Using this physical argument, we have been able to obtain a self-consistent Kolmogorov-like solution for the RG equations. Simple power counting in Eqs. (16) and (20) shows that that $B_0=\mathrm{const}$, and $E(k)\propto k^{-3/2}$ (prediction of the KID phenomenology) does not satisfy the RG equations, indicating that modification in KID's argument is necessary. This paper presents one solution which resolves this problem. In this paper, we have worked out the energy spectrum and renormalized mean magnetic field for $E^{+}=E^{-}$ and $r_A=1$ for simplicity of the calculation. The generalization to arbitrary parameters is planned for future studies.

In our methodology, the averaging has been performed for small wave numbers, in contrast to the earlier RG analysis of turbulence in which higher wavenumbers were averaged out. In our scheme we obtain a self-consistent power-law energy spectrum for large wave number modes, and the spectrum is independent of the small wave number forcing states. This is in agreement with the Kolmogorov's hypothesis, which states that the energy spectrum of the intermediate scale is independent of the large-scale forcing. Any extension of our scheme to fluid turbulence in presence of large-scale shear, etc, will yield interesting insights into the connection of energy spectrum with large-scale forcing.

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