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FIELD THEORETIC CALCULATION OF SCALAR TURBULENCE

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The cascade rate of passive scalar and Bachelor's constant in scalar turbulence are calculated using the flux formula. This calculation is done to first order in perturbation series. Batchelor's constant in three dimension is found to be approximately 1.25. In higher dimension, the constant increases as $d^{1/3}$.

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1. Introduction

Perturbative field-theoretic techniques have been very useful in turbulence research. One of the celebrated field-theoretic method, renormalization groups (RG), has been applied to fluid turbulence,[1−3] scalar turbulence,[2,4] MHD turbulence[5] etc. In RG analysis, one can calculate the renormalized parameters at large length scales. In addition to RG, one can also apply the field-theoretic techniques to calculate turbulent cascade rates.[5,6] In this paper we will calculate the cascade rates of passive scalar using perturbative technique. From this calculation we can also calculate Batchelor's constant, which is very important for large-eddy simulations.

The study of passive scalar is one of the important areas in turbulence research. It finds application in evolution of temperature field, pollution diffusion, etc. The phenomenology of passive scalar is well developed,[6] and their predictions are in agreement with the experimental results. According to the phenomenology, the energy spectrum of both velocity field $\mathbf{u}$ and scalar field ψ in the inertial-convective range are proportional to $k^{-5/3}$. Note that in the inertial-convective range both the nonlinear terms $\mathbf{u} \cdot \nabla \mathbf{u}$ and $\mathbf{u} \cdot \nabla \psi$ dominate the viscous term. However, there exist two other ranges depending on the value of Prandtl number (the ratio of viscosity and diffusivity). In this paper we will only focus on inertial-convective range.

Regarding the calculation of renormalized viscosity and diffusivity for passive scalar admixture, Yakhot and Orszag[2] adopted ϵ-expansion, while Zhou and Vahala,[4] and Lin *et al.*'s[7] procedure is recursive based on the original idea of McComb and his group (Ref. 3 and reference therein). Adzhemyan *et al.*[8] used De

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Dominicis and Martin's[9] procedure for fluid turbulence to passive scalars and computed the renormalized parameters. Earlier, Wyld[10] had given a perturbative expansion of Navier–Stokes equation. Canuto and Dubovikov,[11] and Canuto *et al.*[12] started with Wyld's formalism and computed the renormalized diffusivity for passive scalar; they also computed Batchelor's constant.

Turbulence cascade rates or fluxes play important role in turbulence calculations. It is a measure of transfer of a certain quantity from inside of a wavenumber sphere to the outside wavenumber sphere. In fluid turbulence, Kraichnan[13] applied direct interaction approximation and calculated the flux. Later, the cascade rates have been calculated by many researchers using various techniques, e.g. Eddy–Damped–Quasi–Normal–Markovian (EDQNM) closure scheme, RG, etc. Here we are interested in the cascade rate of passive scalar. This cascade rate quantifies how scalar fluctuations at large length-scales diffuse to small length-scales.

In this paper we apply perturbative techniques to calculate the cascade rate of passive scalar. In our scheme the cascade rate of passive scalar is calculated using the flux formula and the renormalized parameters. In Sec. 2 we recapitulate the earlier RG calculation[4] and extend their results to higher dimensions. In the subsequent section we apply the perturbative technique and calculate the cascade rate in the inertial-convective range to first order. The final expression involves energy spectrum for which we substitute $k^{-5/3}$ obtained from the phenomenology. From this procedure we also calculate Batchelor's constant. We have extended our calculations to higher space dimensions, because higher-dimensional field theory usually provide important insights into the nature of nonlinear interactions.[14]

The outline of the paper is as follows: in Sec. 2 we provide the definitions and recapitulation of the renormalization procedure for passive scalar. In Sec. 3, we carry out the calculation of flux of passive scalar and Batchelor's constant. Section 4 contains conclusions.

2. Renormalization of Viscosity and Diffusivity Revisited

Earlier calculations of renormalization in scalar turbulence have been carried out by Yakhot and Orszag,[2] Zhou and Vahala,[4] and Lin *et al.*[7] In this section we recapitulate very briefly Zhou and Vahala's calculation for passive scalar and extend their results to higher dimensions. Zhou and Vahala's calculation is based on recursive scheme proposed by McComb and his coworker (Ref. 3 and references therein). The equations for the velocity $\mathbf{u}$ and passive scalar ψ fields in Fourier space are

$$(-i\omega + \nu k^2)u_i(\hat{k}) = -\frac{i}{2}P_{ijm}(\mathbf{k})\int d\hat{p}\, u_j(\hat{p})u_m(\hat{k}-\hat{p}) \tag{2.1}$$

$$(-i\omega + \kappa k^2)\psi(\hat{k}) = -ik_j\int d\hat{p}\, u_j(\hat{p})\psi(\hat{k}-\hat{p}) \tag{2.2}$$

with

$$P_{ijm}(\mathbf{k}) = k_j P_{im}(\mathbf{k}) - k_m P_{ij}(\mathbf{k}); \tag{2.3}$$

$$P_{im}(\mathbf{k}) = \delta_{im} - \frac{k_i k_m}{k^2}; \tag{2.4}$$

$$\hat{k} = (\mathbf{k}, \omega); \tag{2.5}$$

$$d\hat{p} = \frac{d\mathbf{p} d\omega}{(2\pi)^{d+1}}. \tag{2.6}$$

Here ν and κ are the viscosity and diffusivity respectively, p is the fluid pressure, and d is the space dimension. We have assumed that the flow is incompressible, i.e. $k_i u_i(\mathbf{k}) = 0$.

In the recursive RG procedure the wavenumber range (k_N, k_0) is divided logarithmically into N shells. The effective parameters are obtained by eliminating the high wavenumber shells iteratively. We denote the higher wavenumber shells by $k^>$ and the remaining wavenumber region by $k^<$. In this procedure the field variables $u_i^>(\hat{k})$ and $\psi^>(\hat{k})$ are assumed to be Gaussian with zero mean, and

$$\langle u_i^>(\hat{p}) u_j^>(\hat{q}) \rangle = P_{ij}(\mathbf{p}) C^u(\hat{p}) \delta(\hat{p} + \hat{q}) \tag{2.7}$$

$$\langle \psi^>(\hat{p}) \psi^>(\hat{q}) \rangle = C^\psi(\hat{p}) \delta(\hat{p} + \hat{q}) \tag{2.8}$$

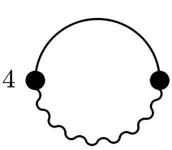
where $C^u(\hat{p})$ and $C^\psi(\hat{p})$ are velocity and scalar correlation functions respectively.

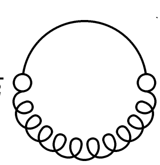
If we denote $\nu_{(n)}$ and $\kappa_{(n)}$ as viscosity and diffusivity respectively after the elimination of n shells, then the elimination of the next shell yields the following equations to the first order in perturbation:

$$(-i\omega + \nu_{(n)} k^2 + \delta\nu_{(n)} k^2) u_i^<(\hat{k}) = -\frac{i}{2} P_{ijm}(\mathbf{k}) \int d\hat{p}\, u_j^<(\hat{p}) u_m^<(\hat{k} - \hat{p}) \tag{2.9}$$

$$(-i\omega + \kappa_{(n)} k^2 + \delta\kappa_{(n)} k^2) \psi^<(\hat{k}) = -ik_j \int d\hat{p}\, u_j^<(\hat{p}) \psi^<(\hat{k} - \hat{p}) \tag{2.10}$$

where

$$\delta\nu_{(n)}(k) = -\frac{1}{(d-1)k^2} \times 4$$  (2.11)

$$\delta\kappa_{(n)}(k) = -\frac{1}{k^2}$$  . (2.12)

In the above Feynmann diagrams, the solid, wiggly (photon), and curly (gluon) lines represent correlation function $\langle u_i u_j \rangle$, and Green functions G^u, G^ψ respectively. The filled circle represents $(-i/2)P_{ijm}$ vertex, while the empty circle represents $-ik_j$

vertex. The RG procedure adopted here is the same as that of Zhou and Vahala.[4] Some of the notation used here is close to the that of MHD turbulence calculation of Verma.[5,15]

The frequency dependence of the correlation function are taken as: $C^u(k,\omega) = 2C^u(k)\Re(G^u(k,\omega))$ and $C^\psi(k,\omega) = 2C^\psi(k)\Re(G^\psi(k,\omega))$. With this assumption, the expressions corresponding to the above Feynmann diagrams will be

$$\delta\nu_{(n)}(k) = \frac{1}{(d-1)k^2}\int_{\mathbf{p}+\mathbf{q}=\mathbf{k}}^{\Delta} \frac{d\mathbf{p}}{(2\pi)^d} S_1(k,p,q) \frac{C^u(q)}{\nu_{(n)}(p)p^2 + \nu_{(n)}(q)q^2}\,, \tag{2.13}$$

$$\delta\kappa_{(n)}(k) = \frac{1}{k^2}\int_{\mathbf{p}+\mathbf{q}=\mathbf{k}}^{\Delta} \frac{d\mathbf{p}}{(2\pi)^d} S_2(k,p,q) \frac{C^u(q)}{\kappa_{(n)}(p)p^2 + \nu_{(n)}(q)q^2}\,, \tag{2.14}$$

with

$$S_1(k,p,q) = kp[(d-3)z + 2z^3 + (d-1)xy]\,, \tag{2.15}$$

$$S_2(k,p,q) = kp(z + xy)\,, \tag{2.16}$$

The quantities x, y, and z are defined by

$$x = -\frac{\mathbf{p}\cdot\mathbf{q}}{pq}; \qquad y = \frac{\mathbf{q}\cdot\mathbf{k}}{qk}; \qquad z = \frac{\mathbf{p}\cdot\mathbf{k}}{pk}\,. \tag{2.17}$$

The effective viscosity and diffusivity after the elimination of $(n+1)$ shell are

$$(\nu,\kappa)_{(n+1)}(k) = (\nu,\kappa)_{(n)}(k) + \delta(\nu,\kappa)_{(n)}(k)\,. \tag{2.18}$$

The spectrum $C^u(k)$ can be written in terms of one-dimensional energy spectrum $E^u(k)$ as

$$C^u(k) = \frac{2(2\pi)^d}{S_d(d-1)} k^{-(d-1)} E^u(k)\,, \tag{2.19}$$

where S_d is the surface area of d dimensional spheres. It is known that $E^u(k)$ follows Kolmogorov's spectrum, i.e.,

$$E^u(k) = K^u(\Pi^u)^{2/3}k^{-5/3}\,, \tag{2.20}$$

where Π is the kinetic-energy flux, and K^u is Kolmogorov's constant for fluid turbulence. Using the dimensional arguments we find that $\nu_{(n)}$ and $\kappa_{(n)}$ have the following forms:

$$(\nu,\kappa)_{(n)}(k_n k') = (K^u)^{1/2}(\Pi^u)^{1/3}k_n^{-4/3}(\nu,\kappa)^*_{(n)}(k')\,, \tag{2.21}$$

with $k = k_{n+1}k'(k' < 1)$. The large-n limit of the $\nu^*_{(n)}(k')$ and $\kappa^*_{(n)}(k')$ are expected to be universal functions in the RG sense.

We solve for $\nu^*_{(n)}(k')$ and $\kappa^*_{(n)}(k')$ iteratively using Eqs. (2.13), (2.14) and (2.18). We take $h = 0.7$, and start with constant $\nu^*_{(0)}$ and $\kappa^*_{(0)}$. We iterate the process till $\nu^*_{(n+1)}(k') \approx \nu^*_{(n)}(k')$ and $\kappa^*_{(n+1)}(k') \approx \kappa^*_{(n)}(k')$, that is, till they converge. We find

Table 1. The computed values of renormalized viscosity ν^*, diffusivity κ^*, turbulent Prandtl number Pr_{turb}, Kolmogorov's constant K^u and Batchelor's constants K^ψ for various space dimensions d.

d	ν^*	κ^*	Pr_{turb}	K^u	K^ψ
3	0.36	0.85	0.42	1.53	1.25
4	0.42	0.69	0.61	1.60	1.39
7	0.38	0.48	0.80	1.76	1.65
10	0.34	0.39	0.87	1.94	1.83
25	0.22	0.24	0.94	2.43	2.44
50	0.16	0.16	1.0	3.1	3.0
100	0.093	0.095	0.98	3.4	3.4

that the iteration process converges; the limiting value ν^* and κ^* are shown in Table 1.

We can draw many interesting conclusions from the above results. Since the scalar does not appear in the equation for u, ν^* computed here is the same as that obtained for fluid turbulence. In Table 1 we have listed the renormalized diffusivity κ^* and the turbulent Prandtl number Pr_{turb}. For $d = 3$, $\kappa^* = 0.85$ and $Pr_{\text{turb}} = \nu^*/\kappa^* = 0.42$. The above quantities vary a bit with the variation of h, but they are roughly in the same range. The error in our estimate of the parameters is of the order of 0.1. Our results are in the same range as those obtained by Zhou and Vahala.[4]

We have also carried out the above analysis for higher space dimensions. The calculated κ^* and Pr_{turb} are listed in Table 1. For large d, $\nu^* \approx \kappa^* \propto d^{-1/2}$. The d dependence is in the agreement with the finding of Fournier and Frisch for fluid turbulence.[16] The above result also implies that $Pr_{\text{turb}} \approx 1$ for large d.

In two dimensions the scalars are not constrained to double energy-enstrophy conservation like velocity field. The RG analysis for two-dimensional scalar turbulence is beyond the scope of this paper.

3. Calculation of Cascade Rates

In this section we compute cascade rates of u and ψ, and Bachelor's constant. To this end we use the flux formulas and the renormalized parameters computed in the previous section. The time evolution of correlation functions C^u and C^ψ (defined by Eqs. (2.7) and (2.8)) are given by[6,17–19]

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) C^u(\mathbf{k}, t, t) = \frac{1}{(d-1)(2\pi)^d \delta(\mathbf{k} + \mathbf{k}')} \times \int_{\mathbf{k}'+\mathbf{p}+\mathbf{q}=\mathbf{0}} \frac{d\mathbf{p}}{(2\pi)^d} [S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + S^{uu}(\mathbf{k}'|\mathbf{q}|\mathbf{p})] \quad (3.1)$$

$$\left(\frac{\partial}{\partial t}+2\kappa k^2\right)C^{\psi}(\mathbf{k},t,t)=\frac{1}{(2\pi)^d\delta(\mathbf{k}+\mathbf{k}')}\times\int_{\mathbf{k}'+\mathbf{p}+\mathbf{q}=\mathbf{0}}\frac{d\mathbf{p}}{(2\pi)^d}[S^{\psi\psi}(\mathbf{k}'|\mathbf{p}|\mathbf{q})+S^{\psi\psi}(\mathbf{k}'|\mathbf{q}|\mathbf{p})] \quad (3.2)$$

where

$$S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})=-\Im([\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})][\mathbf{u}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{p})])\,, \quad (3.3)$$

$$S^{\psi\psi}(\mathbf{k}'|\mathbf{p}|\mathbf{q})=-\Im([\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})][\psi(k')\psi(p)])\,. \quad (3.4)$$

Here $\Im$ stands for the imaginary part of the argument. Note that Eqs. (3.1) and (3.2) have been discussed in the earlier literature, e.g., Lesieur[6] and Stanišić.[17] However, reinterpretation of the terms $S(\mathbf{k}|\mathbf{p}|\mathbf{q})$ by Dar *et al.*[19] as energy transfer from mode $\mathbf{p}$ (the second argument of S) to $\mathbf{k}$ (the first argument of S) with mode $\mathbf{q}$ (the third argument of S) as a mediator makes the formalism more transparent and simple. Also, some quantities which were impossible to calculate in earlier formalism could be computed now.[19] This interpretation of Dar *et al.* is consistent with the earlier formalism.

The energy fluxes Π^u and Π^ψ from a wavenumber sphere of radius k_0 is[19]

$$\Pi^u(k_0)=\int_{k'>k_0}\frac{d\mathbf{k}'}{(2\pi)^d}\int_{p<k_0}\frac{d\mathbf{p}}{(2\pi)^d}\langle S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})\rangle\,, \quad (3.5)$$

$$\Pi^\psi(k_0)=\int_{k'>k_0}\frac{d\mathbf{k}'}{(2\pi)^d}\int_{p<k_0}\frac{d\mathbf{p}}{(2\pi)^d}\langle S^{\psi\psi}(\mathbf{k}'|\mathbf{p}|\mathbf{q})\rangle\,. \quad (3.6)$$

Note that there is no cross-transfer between u and ψ energy. It is also important to note that both C^u and C^ψ are conserved in every triad interaction, i.e.,

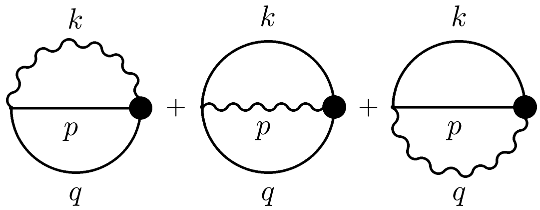
$$S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})+S^{uu}(\mathbf{k}'|\mathbf{q}|\mathbf{p})+S^{uu}(\mathbf{p}|\mathbf{k}'|\mathbf{q})+S^{uu}(\mathbf{p}|\mathbf{q}|\mathbf{k}')+S^{uu}(\mathbf{q}|\mathbf{k}'|\mathbf{p})+S^{uu}(\mathbf{q}|\mathbf{p}|\mathbf{k}')=0 \quad (3.7)$$

$$S^{\psi\psi}(\mathbf{k}'|\mathbf{p}|\mathbf{q})+S^{\psi\psi}(\mathbf{k}'|\mathbf{q}|\mathbf{p})+S^{\psi\psi}(\mathbf{p}|\mathbf{k}'|\mathbf{q})+S^{\psi\psi}(\mathbf{p}|\mathbf{q}|\mathbf{k}')+S^{\psi\psi}(\mathbf{q}|\mathbf{k}'|\mathbf{p})+S^{\psi\psi}(\mathbf{q}|\mathbf{p}|\mathbf{k}')=0\,. \quad (3.8)$$

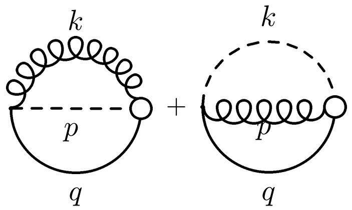
These are the statements of "detailed conservation of energy" in triad interaction (when $\nu=\kappa=0$).[6]

The energy fluxes can be calculated using Eqs. (3.5) and (3.6) by taking ensemble averages of S^{uu} and $S^{\psi\psi}$. It is easy to check that $\langle S^{uu}\rangle=\langle S^{\psi\psi}\rangle=0$ to the zeroth order, but are nonzero to the first order. The field-theoretic calculation performed here is very similar to Verma's MHD flux calculation.[18] Please refer to Verma's paper[18] for further details. The Feynmann diagrams for the first order of $\langle S \rangle$ are

$$\langle S^{uu}(k'|p|q)\rangle = \quad (3.9)$$



$$\langle S^{\psi\psi}(k'|p|q)\rangle = \quad (3.10)$$



In the above Feynmann diagrams, the solid, dashed, wiggly (photon), and curly (gluon) lines represent $\langle u_i u_j \rangle$, $\langle \psi\psi \rangle$, G^u, and G^ψ respectively. In all the diagrams, the left vertex denotes k_i, while the filled circle and the empty circles of right vertex represent $(-i/2)P_{ijm}$ and $-ik_j$ respectively. Algebraically,

$$\begin{aligned}\langle S^{uu}(k|p|q)\rangle = \int_{-\infty}^{t} dt'[&T_1(k,p,q)G^u(k,t-t')C^u(p,t,t')C^u(q,t,t') \\ &+ T_2(k,p,q)G^u(p,t-t')C^u(k,t,t')C^u(q,t,t') \\ &+ T_3(k,p,q)G^u(q,t-t')C^u(k,t,t')C^u(p,t,t')] \,. \end{aligned} \quad (3.11)$$

$$\begin{aligned}\langle S^{\psi\psi}(k|p|q)\rangle = \int_{-\infty}^{t} dt'[&T_4(k,p,q)G^\psi(k,t-t')C^\psi(p,t,t')C^u(q,t,t') \\ &+ T_5(k,p,q)G^\psi(p,t-t')C^\psi(k,t,t')C^u(q,t,t')] \,, \end{aligned} \quad (3.12)$$

where $T_i(k,p,q)$'s are given by

$$T_1(k,p,q) = -kp((d-3)z + (d-2)xy + 2z^3 + 2xyz^2 + x^2 z)\,, \quad (3.13)$$

$$T_2(k,p,q) = kp((d-3)z + (d-2)xy + 2z^3 + 2xyz^2 + y^2 z)\,, \quad (3.14)$$

$$T_3(k,p,q) = kq(xz - 2xy^2 z - yz^2)\,, \quad (3.15)$$

$$T_4(k,p,q) = k^2(1-y^2)\,, \quad (3.16)$$

$$T_5(k,p,q) = -kp(z+xy)\,. \quad (3.17)$$

We assume the relaxation time for $C^u(k)$ and $C^\psi(k)$ to be $(\nu(k)k^2)^{-1}$ and $(\kappa(k)k^2)^{-1}$ respectively, i.e.

$$C^u(k,t,t') = \exp(-\nu(k)k^2(t-t'))C^u(k,t,t)\,, \quad (3.18)$$

$$C^\psi(k,t,t') = \exp(-\kappa(k)k^2(t-t'))C^\psi(k,t,t)\,. \quad (3.19)$$

With this assumption, Eqs. (3.11) and (3.12) reduce to

$$\Pi^u(k_0) = \int_{k>k_0} \frac{d\mathbf{k}}{(2\pi)^d} \int_{p<k_0} \frac{d\mathbf{p}}{(2\pi)^d} \frac{1}{\nu(k)k^2 + \nu(p)p^2 + \nu(q)q^2} \times [T_1(k,p,q)C^u(p)C^u(q) + T_2(k,p,q)C^u(k)C^u(q) + T_3(k,p,q)C^u(k)C^u(p)] \tag{3.20}$$

$$\Pi^\psi(k_0) = \int_{k>k_0} \frac{d\mathbf{k}}{(2\pi)^d} \int_{p<k_0} \frac{d\mathbf{p}}{(2\pi)^d} \frac{1}{\kappa(k)k^2 + \kappa(p)p^2 + \nu(q)q^2} \times [T_4(k,p,q)C^\psi(p)C^u(q) + T_5(k,p,q)C^\psi(k)C^u(q)]\,. \tag{3.21}$$

For $C^u(k)$ we substitute Eqs. (2.19) and (2.20), while for C^ψ we substitute[6]

$$C^\psi(k) = \frac{2(2\pi)^d}{S_d} k^{-(d-1)} E^u(k)\,, \tag{3.22}$$

$$E^\psi(k) = K^\psi \Pi^\psi (\Pi^u)^{-1/3} k^{-5/3}\,, \tag{3.23}$$

where K^ψ is called the Batchelor's constant. The renormalized viscosity and diffusivity in the inertial range are

$$\nu(k) = (K^u)^{1/2} (\Pi^u)^{1/3} k^{-4/3} \nu^* \tag{3.24}$$

$$\kappa(k) = (K^u)^{1/2} (\Pi^u)^{1/3} k^{-4/3} \kappa^*\,. \tag{3.25}$$

The substitution of the above quantities, and the change of variables

$$k = \frac{k_0}{u}; \qquad p = \frac{k_0}{u} v; \qquad q = \frac{k_0}{u} w \tag{3.26}$$

yield the following nondimensional version of the flux equations[16]:

$$1 = (K^u)^{3/2} \left[\frac{4S_{d-1}}{(d-1)^2 S_d} \int_0^1 dv \ln\left(\frac{1}{v}\right) \int_{1-v}^{1+v} dw (vw)^{d-2} (\sin\alpha)^{d-3} F^u(v,w) \right] \tag{3.27}$$

$$1 = K^\psi (K^u)^{1/2} \left[\frac{4S_{d-1}}{(d-1) S_d} \int_0^1 dv \ln\left(\frac{1}{v}\right) \int_{1-v}^{1+v} dw (vw)^{d-2} (\sin\alpha)^{d-3} F^\psi(v,w) \right] \tag{3.28}$$

where α is angle between vectors $\mathbf{p}$ and $\mathbf{q}$, and the integrals $F^{u,\psi}(v,w)$ are

$$F^u = \frac{1}{\nu^*(1 + v^{2/3} + w^{2/3})} [t_1(v,w)(vw)^{-d-\frac{2}{3}} + t_2(v,w) w^{-d-\frac{2}{3}} + t_3(v,w) v^{-d-\frac{2}{3}}] \tag{3.29}$$

$$F^\psi = \frac{1}{\kappa^*(1 + v^{2/3}) + \nu^* w^{2/3}} [t_4(v,w)(vw)^{-d-\frac{2}{3}} + t_5(v,w) w^{-d-\frac{2}{3}}]\,. \tag{3.30}$$

Here $t_i(v,w) = T_i(k, kv, kw)/k^2$.

The terms in the square brackets of Eqs. (3.27) and (3.28) (denoted by $I^{u,\psi}$) involve integrals. We compute them using Gaussian quadrature. The integrals converge for all dimensions $d \geq 2$. Once the integrals are known, Kolmogorov's and Batchelor's constants (K^u and K^ψ respectively) can be computed. The computed values are given in Table 1.

In our calculation Batchelor's constant K^ψ in three dimension is 1.25. Due to uncertainties in the value of ν^* and κ^*, the error in the constant could be of the order of 0.1. Earlier, Kraichnan had estimated the constant to be 0.2. Yakhot and Orszag[2] obtained $K^\psi = 1.16$ by their ϵ-based renormalization group analysis. Canuto and Dubovikov[11] and Canuto *et al.*[12] estimated $K^\psi = (5/3) * 0.72 = 1.2$ using their RG calculation. Lin *et al.*[7] find the constant to be close to 0.3. Our result is in very good agreement with the theoretical predictions of Yakhot and Orszag[2] and Canuto *et al.*,[12] as well as to the experimental values ($\approx 1.2 - 1.4$, see Monin and Yaglom[20]).

It is also interesting to note that both $K^{u,\psi}$ are proportional to $d^{1/3}$, consistent with the predictions of Fournier and Frisch[16] for fluid turbulence. This result implies that the cascade rated $\Pi^{u,\psi}$ will decrease with dimensions as $d^{-1/2}$.

4. Conclusions

In this paper we employed field-theoretic techniques to calculate the cascade rates of scalar turbulence. Our calculation is to first order. From this formalism we also calculate Batchelor's constant. In three dimensions, we find Batchelor's constant to be 1.25, which is in very good agreement with the theoretical predictions of Yakhot and Orszag[2] and Canuto *et al.*,[12] and the experimental values. In higher space dimensions the constant varies as $d^{1/3}$.

Our calculation of cascade rate requires the renormalized viscosity and diffusivity. We have extended the RG calculations of Zhou and Vahala[4] for higher dimensions. Our calculations show that for higher dimensions, the renormalized viscosity and diffusivity vary with dimensions as $d^{-1/2}$, and the turbulent Prandtl number approaches unity.

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